Asymptotics of the Fourier transform of the spectral measure for Schrödinger operators with bounded and unbounded sparse potentials

2002 J. Phys. A: Math. Gen. 35 6393

(http://iopscience.iop.org/0305-4470/35/30/314)

J. Phys. A: Math. Gen. **35** (2002) 6393–6417 PII: S0305-4470(02)35809-8

Asymptotics of the Fourier transform of the spectral measure for Schrödinger operators with bounded and unbounded sparse potentials

Denis Krutikov

Fachbereich Mathematik/Informatik, Universität Essen, 45117 Essen, Germany

E-mail: denis.krutikov@uni-essen.de

Received 10 April 2002
Published 19 July 2002
Online at stacks.iop.org/JPhysA/35/6393

Abstract

We study the pointwise behaviour of the Fourier transform of the spectral measure for discrete one-dimensional Schrödinger operators with unbounded sparse potentials, particularly with the potentials of the special type, which give rise to the spectra with Hausdorff dimensionality between $1/2$ and 1. Operators with bounded sparse potentials are also considered.

PACS numbers: 02.30.Tb, 02.30.Nw, 03.65.−w
Mathematics Subject Classification: 34L40, 81Q10, 42A38

1. Introduction

Let H be a Schrödinger operator acting on a Hilbert space $\mathcal{H}$. We can think of H as an energy Hamiltonian of a quantum-mechanical system. Denote by ψ the initial state of such a system. Then the time evolution of the state $\psi(t)$ is given by $\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi$. One important dynamical property of the system is the probability of finding the system again in the state ψ at time t, which is given by $|\langle\psi, \psi(t)\rangle|^2 = |\langle\psi, \mathrm{e}^{-\mathrm{i}tH}\psi\rangle|^2$. The question of special interest is whether this probability goes to zero for large $|t|$. Such scattering is a typical feature of systems with absolute continuous spectra, which is easy to show using the spectral resolution of the operator H. For this denote by ρ_ψ the spectral measure of ψ. Then we have the representation $\langle\psi, \mathrm{e}^{-\mathrm{i}tH}\psi\rangle = \int \mathrm{e}^{-\mathrm{i}tx}\,\mathrm{d}\rho_\psi(x) = \widehat{\rho}_\psi(t)$ (which is exactly the Fourier transform of the spectral measure ρ_ψ) and we can therefore reduce our discussion to the study of the asymptotic behaviour of $\widehat{\rho}_\psi(t)$ at infinity, particularly, the above-mentioned question is equivalent to the question whether the following relation holds:

$$\lim_{t\to\pm\infty} \widehat{\rho}_\psi(t) = 0. \qquad (1)$$

(Measures for which (1) holds are called Rajchman measures. For more information about these measures, see [4, 6].)

This last question is especially easy to answer if the measure under consideration is an absolutely continuous measure or a point measure. The answer is positive in the first case (by the Riemann–Lebesgue lemma), in particular, the systems with the absolute continuous spectra have the spreading property $\langle \psi, \psi(t) \rangle \to 0$. In the second case the answer is negative (by Wiener's theorem), telling us that bound states do not scatter. So the only complicated (and therefore also interesting) case concerns the singular continuous part of a measure.

In this paper, we will study one specific model, for which the pointwise behaviour of $\widehat{\rho}(t)$ can be analysed completely, although the arising spectrum is singular continuous. Namely, we will consider discrete one-dimensional Schrödinger operators on the 'half line' (that is on $\ell_2(\mathbb{N})$), which are defined by

$$(H_\varphi y)(n) = y(n-1) + y(n+1) + V(n)y(n)$$

(where $0 < \varphi < \pi$) along with a phase boundary condition

$$y(0)\sin\varphi + y(1)\cos\varphi = 0$$

with sparse potentials, that is potentials of the form

$$V(l) = \begin{Bmatrix} v_n & l = x_n \\ 0 & \text{else} \end{Bmatrix} \tag{2}$$

where $1 < x_1 < x_2 < \cdots < x_n < \cdots$ is a rapidly increasing sequence.

To understand why the systems described by such operators can have the spreading property $\widehat{\rho}(t) \to 0$, let us consider the quasiclassical picture of quantum motion under the influence of a sparse potential. We can think of a sparse potential as a sequence of the barriers, which the particle has to meet starting at time $t = 0$ at the origin and moving to the right. When the particle hits each barrier, it is either reflected or transmitted (with the corresponding probabilities). In the case of reflection, the particle returns to the origin, while in the case of transmission, it moves on to the next barrier, where it is again either transmitted or reflected. So we see that $\widehat{\rho}(t)$ (recall that $|\widehat{\rho}(t)|^2$ is the probability of finding the particle again at $n = 1$ at time t if it was initially localized at $n = 1$) should have a resonance structure since return to the origin is possible only at certain times. Because of the spreading of the wave packets, we should not expect very sharp resonances. Moreover, if the potential is sufficiently sparse then the spreading of the wave packets between the barriers should lead to the announced property $|\widehat{\rho}(t)| \to 0$.

That sparse potentials can lead to singular continuous spectra was first shown by Pearson in [7] (for continuous Schrödinger operators). As for the case of discrete operators with unbounded potentials, it was shown by Simon and Stolz in [9] that if x_n grows sufficiently fast then the spectrum in $(-2, 2)$ is purely singular continuous.

Moreover, sparse potentials can be used to obtain Schrödinger operators with spectra with exactly known Hausdorff dimensionality between 0 and 1, as the following theorem shows.

Theorem 1.1 (due to [1], [2]). *Let $\delta \in (0, 1)$. Suppose a potential V is defined by (2), and suppose the sequences (v_n) and (x_n) obey*

$$v_n = x_n^{\frac{1-\delta}{2\delta}} \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{\prod_{k=1}^{n-1} x_k}{x_n^\varepsilon} = 0 \qquad \textit{for all} \quad \varepsilon > 0. \tag{3}$$

Then

(i) *For every boundary phase φ, the spectrum of H_φ consists of the closed interval $[-2, 2]$ (which is the essential spectrum $\sigma_{\text{ess}}(H_\varphi)$) along with some discrete point spectrum outside this interval.*

(ii) For every φ, the Hausdorff dimensionality of the spectrum of H_φ in $(-2, 2)$ is bounded between dimensions δ and $\frac{2\delta}{1+\delta}$.

(iii) For Lebesgue a.e. φ, the spectrum in $[-2, 2]$ is of exact dimension δ.

We will mainly consider in this paper one-dimensional discrete Schrödinger operators on $\ell_2(\mathbb{N})$ under the same assumptions as in this theorem. We are only interested in the part of the spectrum in $(-2, 2)$, because of the relation $\sigma_{\text{ess}}(H_\varphi) = [-2, 2]$. So our aim is to investigate the asymptotics of the Fourier transform of the spectral measure for Schrödinger operators in the situation, where it is not only known that their (essential) spectra are purely singular continuous, but where the exact Hausdorff dimensionality of these spectra is also known. (For other results on the relationship between asymptotics of $\widehat{\rho}(t)$ and the continuity properties of ρ with respect to Hausdorff measures, see [5].)

The actual value φ from the definition of the operator H_φ will not be significant, and all results of this paper are valid for any value φ. Therefore we usually omit the index φ and write H instead of H_φ.

Let ρ be the spectral measure associated with the vector $\delta_1 \in \ell_2$ ($\delta_1(1) = 1$ and $\delta_1(n) = 0$ if $n \neq 1$), that is $\rho(M) = \|E(M)\delta_1\|^2$, where $E(\cdot)$ is the spectral resolution of H. Since δ_1 is a cyclic vector for H, any other spectral measure ρ_ψ is absolutely continuous with respect to ρ. We can therefore restrict our further consideration to this particular measure.

We will prove the following theorem:

Theorem 1.2. *Let a potential V have the form (2) and let the sequences (x_n) and (v_n) obey (3). Then:*

(i) For every $\delta \in \left(\frac{4}{5}, 1\right)$, $f \in C_0^\infty(-2, 2)$ and every $\sigma > 0$, there exists a constant C such that

$$|(f\,\mathrm{d}\rho)\widehat{\ }(t)| \leqslant C|t|^{-\frac{5}{4}+\frac{1}{\delta}+\sigma}$$

for all t with $|t| > 1$.

(ii) For every $\delta \in \left(\frac{2}{3}, 1\right)$, $\sigma > 0$ and every $f \in C_0^\infty(-2, 2)$ with $0 \notin \operatorname{supp} f$, there exists a constant C such that

$$|(f\,\mathrm{d}\rho)\widehat{\ }(t)| \leqslant C|t|^{-\frac{3}{2}+\frac{1}{\delta}+\sigma}$$

for all t with $|t| > 1$.

(iii) Fix arbitrary $\epsilon > 0$ and $\delta \in \left(\frac{1}{2}, 1\right)$ and define the resonant set $\mathcal{R}$ by

$$\mathcal{R} = \bigcup_{n\in\mathbb{N}} \left[\frac{1}{2}x_n, x_n^{\frac{\delta}{2\delta-1}+\varepsilon}\right].$$

Then for every $m \in \mathbb{N}$ and every $f \in C_0^\infty(-2, 2)$, there exist a constant C and $t_0 > 0$ such that

$$|(f\,\mathrm{d}\rho)\widehat{\ }(t)| \leqslant C|t|^{-m}$$

for all t with $|t| \notin \mathcal{R}$ and $|t| > t_0$.

Remark. This result does not actually require the exact relation $v_n = x_n^{\frac{1-\delta}{2\delta}}$. It will be clear from the proof that the precise condition on the sequence (v_n) which we need is that $(|v_n|)$ is an unbounded monotonically increasing sequence, such that $\prod_{r=1}^{j-1} |v_r| \leqslant |v_j|^N$ holds for all j with some $N > 0$ and $v_j \leqslant x_j^{\frac{1-\delta}{2\delta}}$ holds for all j. So our results concern a somewhat more general situation than seems at the first moment.

From theorem 1.2 we see that $\widehat{\rho}$ has the announced resonance structure. We also see that we have the best estimates for the values of δ near to 1, which has the following explanation: the Hausdorff dimension can be seen as a 'measure' of singularity of the spectrum (note that the Hausdorff dimension of the absolute continuous spectrum is equal to 1), so the better the estimates the less 'singular' is the spectrum, that is for δ near to 1.

This paper is closely connected with the paper [4], where discrete one-dimensional Schrödinger operators with bounded sparse potentials are considered. As an illustration of this connection we note that in the limit $\delta \to 1$ the inequality from the last theorem (part (ii)) transforms into the inequality

$$|(f\,\mathrm{d}\rho)\widehat{\ }(t)| \leqslant C|t|^{-\frac{1}{2}+\sigma}$$

which is the inequality proved in [4] in the case of a bounded potential for any f with $0 \notin \operatorname{supp} f$.

We will also briefly discuss the case of a bounded potential at the end of this paper and obtain in the last section as a byproduct of the whole consideration the following theorem:

Theorem 1.3. *Let a potential V have the form (2) with the bounded sequence (v_n) and let the sequence (x_n) obey*

$$\lim_{n\to\infty} \frac{\prod_{k=1}^{n-1} x_k}{x_n^{\varepsilon}} = 0 \qquad \textit{for all} \quad \varepsilon > 0. \tag{4}$$

Then for every $f \in C_0^\infty(-2,2)$ and every $\sigma > 0$, there exists a constant C, such that

$$|(f\,\mathrm{d}\rho)\widehat{\ }(t)| \leqslant C|t|^{-\frac{1}{4}+\sigma}$$

for all t with $|t| > 1$.

This theorem can be considered as some kind of a completion of results from [4], for this theorem provides us with the general estimate for $(f\,\mathrm{d}\rho)\widehat{\ }(t)$ for arbitrary f, whereas in [4] only the corresponding formula for f with $0 \notin \operatorname{supp} f$ is proved (we have, however, to note here that our assumptions on the growth of x_n are much stronger than those from [4]).

Our approach to prove theorems 1.2 and 1.3 depends on a representation of the Fourier transform of the spectral measure as a series of integrals (theorem 2.2). We describe our strategy of estimating the integrals from this series in section 5.

From now on and in the rest of this paper (with the exception of section 9), we assume that the potential is given by (2) and (3) (if nothing else is stated).

2. Preliminaries

We collect in this section some results that we will use in the sequel. First, we will use an EFGP transformation (also called a Prüfer transformation) to rewrite the discrete Schrödinger equation

$$y(n-1) + y(n+1) + V(n)y(n) = Ey(n) \qquad (n \in \mathbb{N}). \tag{5}$$

So, suppose that $E \in (-2, 2)$ and let y be some solution of (5). Write $E = 2\cos x$ with $x \in (0, \pi)$ and define $R(n) > 0, \theta(n)$ by

$$\begin{pmatrix} u(n) - u(n-1)\cos x \\ u(n-1)\sin x \end{pmatrix} = R(n) \begin{pmatrix} \cos(\theta(n)) \\ \sin(\theta(n)) \end{pmatrix}.$$

Then R and θ obey the equations (see [3])

$$\frac{R(n+1)^2}{R(n)^2} = 1 - \frac{V(n)}{\sin x}\sin(2\theta(n) + 2x) + \frac{V(n)^2}{\sin^2 x}\sin^2(\theta(n) + x) \tag{6}$$

$$\cot(\theta(n+1)) = \cot(\theta(n) + x) - \frac{V(n)}{\sin x}. \tag{7}$$

It is evident that for all l from $\{x_n + 1, \ldots, x_{n+1}\}$, $n \in \mathbb{N}$, holds: $R(l) = R(x_n + 1)$ and $\theta(l) = \theta(x_n + 1) + x(l - x_n - 1)$. We further denote $\theta(n) + x$ with $\bar{\theta}(n)$, $\theta(x_r)$ with θ_r and $\bar{\theta}(x_r)$ with $\bar{\theta}_r$.

As a second tool, we need a representation (from [4]) of the spectral measure as a weak star limit of absolutely continuous measures. (This result is related to the similar result for the continuous case from [8].) Let $R(n, x) = R(n)$ correspond to the solution u_φ of (5) with the initial values

$$u_\varphi(0, z) = \cos\varphi \qquad u_\varphi(1, z) = \sin\varphi.$$

Proposition 2.1 (due to [4])**.** *Let f be a continuous function with the support contained in $(-2, 2)$. Then*

$$\int_{-2}^{2} f(E)\, \mathrm{d}\rho(E) = \frac{2}{\pi}\lim_{n\to\infty}\int_0^\pi f(2\cos x)\frac{\sin^2 x}{R^2(n, x)}\,\mathrm{d}x.$$

We can use proposition 2.1 to derive a series representation for the Fourier transform of ρ. Since we are only interested in the part of the spectrum in $(-2, 2)$, we will study

$$(f\,\mathrm{d}\rho)\widehat{\ }(t) = \int_{-\infty}^{\infty} f(E)\,\mathrm{e}^{-\mathrm{i}tE}\,\mathrm{d}\rho(E)$$

with $f \in C_0^\infty(-2, 2)$.

Theorem 2.2. *Let the coefficients $C_j^\alpha(x)$ be defined by*

$$C_j^\alpha(x) = \prod_{r=1}^{j}\left(\frac{v_r}{v_r + 2\mathrm{i}(\mathrm{sgn}(\alpha_r))\sin x}\right)^{|\alpha_r|} \tag{8}$$

with $\alpha = (\alpha_1, \ldots, \alpha_j)$ from $\mathbb{Z}^j$ (sgn denotes here a sign). Let f be a continuous function with the support contained in $(-2, 2)$. Let a and b be defined by

$$a = \inf\{x \in (0, \pi) : 2\cos x \in supp\,(f)\} \qquad b = \sup\{x \in (0, \pi) : 2\cos x \in supp\,(f)\}.$$

Then there exists a function $h(x)$ from $C_0^\infty(0, \pi)$ with the support contained in [a,b], such that for any $m \geqslant 1$ holds

$$\begin{aligned}(f\,\mathrm{d}\rho)\widehat{\ }(t) = &\sum_{j=m+1}^{\infty}\sum_{\{\alpha\in\mathbb{Z}^j\,|\alpha_j\neq 0\}}\int_a^b h(x)C_j^\alpha(x)\exp\left(\mathrm{i}\,2\left(-t\cos(x) + \sum_{r=1}^{j}(\alpha_r\bar{\theta}_r)\right)\right)\mathrm{d}x \\ &+ \sum_{\alpha\in\mathbb{Z}^m}\int_a^b h(x)C_m^\alpha(x)\exp\left(\mathrm{i}\,2\left(-t\cos(x) + \sum_{r=1}^{m}(\alpha_r\bar{\theta}_r)\right)\right)\mathrm{d}x.\end{aligned} \tag{9}$$

Proof. From (2) (which ensures that $V(l) = 0$, if $l \neq x_n$) together with (6) we have the representation

$$R^{-2}(x_j, k) = R^{-2}(0, k) \prod_{r=1}^{j-1} \left(1 - \frac{v_r \sin(2\bar{\theta}_r)}{\sin(x)} + \frac{v_r^2 \sin^2(\bar{\theta}_r)}{\sin^2(x)}\right)^{-1}. \tag{10}$$

For any real u the function $\left(1 - u \sin x + u^2 \sin^2 \frac{x}{2}\right)^{-1}$ expands in a Fourier series

$$\left(1 - u \sin x + u^2 \sin^2 \frac{x}{2}\right)^{-1} = \sum_{n=-\infty}^{\infty} \left(\frac{u}{u + 2\mathrm{i}(\mathrm{sgn}(n))}\right)^{|n|} \mathrm{e}^{\mathrm{i}nx}. \tag{11}$$

(This relation can be easily proved by summing the series on the right-hand side.) We use formulae (10) and (11) and proposition 2.1 (where we take the limit with respect to the subsequence x_j) to obtain

$$(f\,\mathrm{d}\rho)\widehat{\ }(t) = \lim_{j\to+\infty} \sum_{\alpha\in\mathbb{Z}^j} \int_a^b h(x) C_j^\alpha(x) \exp\left(\mathrm{i}\,2\left(-t\cos(x) + \sum_{r=1}^{j}(\alpha_r\bar{\theta}_r)\right)\right)\mathrm{d}x$$

where the function h is defined by $h(x) = \frac{2}{\pi} R^{-2}(0, x) f(2\cos x)\sin^2 x$. Then (9) follows from

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{Z}^j} \int_a^b h(x) C_j^\alpha(x) \exp\left(\mathrm{i}\,2\left(-t\cos(x) + \sum_{r=1}^{j}(\alpha_r\bar{\theta}_r)\right)\right)\mathrm{d}x \\
&\quad= \sum_{\{\alpha\in\mathbb{Z}^j|\alpha_j\neq 0\}} \int_a^b h(x) C_j^\alpha(x) \exp\left(\mathrm{i}\,2\left(-t\cos(x) + \sum_{r=1}^{j}(\alpha_r\bar{\theta}_r)\right)\right)\mathrm{d}x \\
&\qquad+ \sum_{\alpha\in\mathbb{Z}^{j-1}} \int_a^b h(x) C_{j-1}^\alpha(x) \exp\left(\mathrm{i}\,2\left(-t\cos(x) + \sum_{r=1}^{j-1}(\alpha_r\bar{\theta}_r)\right)\right)\mathrm{d}x
\end{aligned}$$

which is an easy corollary of $C_j^{(\alpha_1,\ldots,\alpha_{j-1},0)} = C_{j-1}^{(\alpha_1,\ldots,\alpha_{j-1})}$. □

We will use formula (9) in the case $m = m(t)$, where $m(t)$ is defined by the condition

$$\frac{x_{m(t)+1}}{2} \leqslant |t| < \frac{x_{m(t)+2}}{2}. \tag{12}$$

Before we go on, we make some general remarks on the notation.

(i) The interval $[a, b]$ from theorem 2.2 is from now on fixed. We denote $a_0 = \inf_{x\in[a,b]} \sin x$ and use a_0 further in this meaning.

(ii) The term 'constant' will always refer to a positive number which is independent of t, j, x and of the α_j. It may depend, however, on the other parameters of the problem, which are the sequences (v_n) and (x_n) and the function $f \in C_0^\infty(-2, 2)$. It may also depend on additional parameters we introduce such as the σ from theorem 1.2. Some constants will have the same meaning throughout the paper (for example C_k from lemma 4.1 below) or through a certain part of the paper (for example G_l from lemma 5.1 below). In the last case we will always mention the change of this meaning. All other constants, whose actual value may change from one formula to the next, are usually denoted by C. Also, we sometimes write $a \lesssim b$ instead of $a \leqslant Cb$.

(iii) We use the notation $\mathbb{Z}_+$ for the set $\{0, 1, 2, \ldots\}$.

(iv) We always denote the derivative of u (w.r.t. x) of order b by $u^{(b)}$, that is $u^{(b)} = \frac{\mathrm{d}^b}{\mathrm{d}x^b}u$.

3. Estimates on the derivatives of $C_j^\alpha(x)$

The aim of this section is to prove the following lemma:

Lemma 3.1. *Let the sequence (v_n) of positive numbers converge monotonically to $+\infty$. Denote $\frac{\sqrt{v_r^2+4a_0^2}}{v_r}$ by $p_{0,r}$. Then there exists for each integer $k \geqslant 0$ a constant $\tilde{P}_k$, such that for any j*

$$\sum_{\alpha\in\mathbb{Z}^j} \sup_{\substack{\kappa=0,\ldots,k\\ x\in(a,b)}} \left|\left(C_j^\alpha(x)\right)^{(\kappa)}\right| \leqslant \frac{\tilde{P}_k}{(\ln(p_{0,j}))^k} \prod_{r=1}^{j}\left(1+\frac{2}{\sqrt{p_{0,r}}-1}\right). \tag{13}$$

(We prove this lemma at the end of the section.)

We have for the coefficients $C_j^\alpha(x)$ the representation

$$C_j^\alpha = \prod_{r=1}^{j}\left(\frac{v_r}{\sqrt{v_r^2+4a_0^2}}\right)^{|\alpha_r|} \prod_{r=1}^{j}\left(\frac{1}{\frac{v_r}{\sqrt{v_r^2+4a_0^2}}+\frac{2\mathrm{i}(\mathrm{sgn}(\alpha_r))\sin x}{\sqrt{v_r^2+4a_0^2}}}\right)^{|\alpha_r|}. \tag{14}$$

We need therefore only to investigate the derivatives of

$$\prod_{r=1}^{j}\left(\frac{1}{\frac{v_r}{\sqrt{v_r^2+4a_0^2}}+\frac{2\mathrm{i}(\mathrm{sgn}(\alpha_r))\sin x}{\sqrt{v_r^2+4a_0^2}}}\right)^{|\alpha_r|}.$$

We study for the moment a more general situation.

Lemma 3.2. *Let w and s be the real numbers, such that $w^2+4s^2a_0^2=1$ holds. Let m be a non-negative integer. Then there exists for any non-negative integer p a constant $C_{2,p}$, independent of m, w and s, such that for $k=0,\ldots,p$ holds the inequality*

$$\sup_{x\in(a,b)}\left|\left(\left(\frac{1}{w+2\mathrm{i}s\sin x}\right)^m\right)^{(k)}\right| \leqslant C_{2,p}^k m^k.$$

Proof. It is easy to see that for each k there exists a constant $C_{1,k}$, independent of w and s, such that

$$\sup_{x\in(a,b)}\left|\left(\frac{1}{w+2\mathrm{i}s\sin x}\right)^{(k)}\right| \leqslant C_{1,k} \tag{15}$$

holds, where for $k=0$ we can set $C_{1,0}=1$. We use further the common equality

$$\left(\prod_{r=1}^{n}u_r\right)^{(k)} = \sum_{l_1=0}^{k}\cdots\sum_{l_{n-1}=0}^{l_{n-2}}\binom{k}{l_1}\cdots\binom{l_{n-2}}{l_{n-1}}u_1^{(l_{n-1})}u_2^{(l_{n-2}-l_{n-1})}\cdots u_n^{(k-l_1)} \tag{16}$$

and two formulae following from this equality

$$\sum_{l_1=0}^{k}\sum_{l_2=0}^{l_1}\cdots\sum_{l_{n-1}=0}^{l_{n-2}}\binom{k}{l_1}\binom{l_1}{l_2}\cdots\binom{l_{n-2}}{l_{n-1}} = n^k \tag{17}$$

(with $u_r(x)=\mathrm{e}^x$) and

$$\sum_{l_1=0}^{k}\cdots\sum_{l_{n-1}=0}^{l_{n-2}}\binom{k}{l_1}\cdots\binom{l_{n-2}}{l_{n-1}}m_1^{l_{n-1}}\cdots m_n^{k-l_1} = \left(\sum_{r=1}^{n}m_r\right)^k \tag{18}$$

(with $u_r(x) = \mathrm{e}^{m_r x}$). Then with $u_l(x) = \frac{1}{w+2\mathrm{i}s\sin x}$ for $l = 1, \ldots, m$, and

$$C_{2,p} = \max_{l=1,\ldots,p} C_{1,p}$$

from (15)–(17) follow the estimates

$$\left|\left(\left(\frac{1}{w+2\mathrm{i}s\sin x}\right)^m\right)^{(k)}\right| \leqslant \sum_{l_1=0}^{k}\cdots\sum_{l_{m-1}=0}^{l_{m-2}}\binom{k}{l_1}\cdots\binom{l_{m-2}}{l_{m-1}}\left|u_1^{(l_{m-1})}\right|\left|u_2^{(l_{m-2}-l_{m-1})}\right|\cdots\left|u_m^{(k-l_1)}\right|$$

$$\leqslant \sum_{l_1=0}^{k}\cdots\sum_{l_{m-1}=0}^{l_{m-2}}\binom{k}{l_1}\cdots\binom{l_{m-2}}{l_{m-1}}C_{1,l_{m-1}}C_{1,l_{m-2}-l_{m-1}}\cdots C_{1,k-l_1}$$

$$\leqslant C_{2,p}^k\sum_{l_1=0}^{k}\sum_{l_2=0}^{l_1}\cdots\sum_{l_{m-1}=0}^{l_{m-2}}\binom{k}{l_1}\binom{l_1}{l_2}\cdots\binom{l_{m-2}}{l_{m-1}} = C_{2,p}^k m^k$$

where we have used the relations $k = k - l_1 + l_{m-1} + \sum_{j=1}^{m-2}(l_{m-j-1} - l_{m-j})$ and $C_{1,0} = 1$. □

Lemma 3.3. *Let the real numbers w_l and s_l, $l = 1, \ldots, j$, obey $w_l^2 + 4s_l^2a_0^2 = 1$ for all l's. Let m_l, $l = 1, \ldots, j$, be any non-negative integers. Then there exists for each non-negative integer p a constant $C_{2,p}$, independent of w_l, s_l and m_l, such that for all $k = 0, \ldots, p$ holds*

$$\sup_{x\in(a,b)}\left|\left(\prod_{l=1}^{j}\left(\frac{1}{w_l+2\mathrm{i}s_l\sin x}\right)^{m_l}\right)^{(k)}\right| \leqslant C_{2,p}^k\left(\sum_{l=1}^{j}m_l\right)^k.$$

Proof. We use formula (16) with $u_l = \left(\frac{1}{w_l+2\mathrm{i}s_l\sin x}\right)^{m_l}$, $l = 1, \ldots, j$, to obtain, using the result of lemma 3.2 and (18), the following estimates:

$$\sup_{x\in(a,b)}\left|\left(\prod_{l=1}^{j}\left(\frac{1}{w_l+2\mathrm{i}s_l\sin x}\right)^{m_l}\right)^{(k)}\right|$$

$$\leqslant \sum_{l_1=0}^{k}\cdots\sum_{l_{j-1}=0}^{l_{j-2}}\binom{k}{l_1}\cdots\binom{l_{j-2}}{l_{j-1}}\sup_{x\in(a,b)}\left|u_1^{(l_{j-1}-1)}\right|\cdots\sup_{x\in(a,b)}\left|u_j^{(k-l_1)}\right|$$

$$\leqslant \sum_{l_1=0}^{k}\cdots\sum_{l_{j-1}=0}^{l_{j-2}}\binom{k}{l_1}\cdots\binom{l_{j-2}}{l_{j-1}}C_{2,l_{j-1}}^{l_{j-1}}(m_1)^{l_{j-1}}\cdots C_{2,k-l_1}^{k-l_1}(m_j)^{k-l_1}$$

$$\leqslant C_{2,p}^k\sum_{l_1=0}^{k}\cdots\sum_{l_{j-1}=0}^{l_{j-2}}\binom{k}{l_1}\cdots\binom{l_{j-2}}{l_{j-1}}(m_1)^{l_{j-1}}\cdots(m_j)^{k-l_1} = C_{2,p}^k\left(\sum_{l=1}^{j}m_l\right)^k. \qquad \square$$

Corollary 3.4. *For all $k = 0, \ldots, p$ holds the inequality*

$$\sup_{x\in(a,b)}\left(\prod_{r=1}^{j}\left(\frac{1}{\frac{v_r}{\sqrt{v_r^2+4a_0^2}}+\frac{2\mathrm{i}(\mathrm{sgn}(\alpha_r))\sin x}{\sqrt{v_r^2+4a_0^2}}}\right)^{|\alpha_r|}\right)^{(k)} \leqslant C_{2,p}^k\left(\sum_{r=1}^{j}|\alpha_r|\right)^k.$$

From this last inequality and from the formula (14) follows the inequality

$$\sup_{\substack{\kappa=0,\ldots,k\\ x\in(a,b)}}\left|\left(C_j^\alpha(x)\right)^{(\kappa)}\right| \leqslant \frac{C_{2,k}^k\left(\sum_{r=1}^{j}|\alpha_r|\right)^k}{\prod_{r=1}^{j}p_{0,r}^{|\alpha_r|}}. \tag{19}$$

We are now able to prove lemma 3.1.

Proof of lemma 3.1. Denote $\left(\frac{2k}{e}\right)^k$ with $\tilde{C}_k$. The function $\frac{x^k}{a^x}$ with $a > 1$ obeys $\sup_{x\geqslant 0} \frac{x^k}{a^x} \leqslant \frac{k^k}{e^k(\ln a)^k}$. Therefore, for all positive integers n_1 and all $p > 1$, $n_1^k \leqslant \tilde{C}_k \ln(p)^{-k}(\sqrt{p})^{n_1}$ holds. We have then for all positive integers j the following estimates:

$$\sum_{\alpha\in\mathbb{Z}^j}\left(\left(\sum_{r=1}^{j}|\alpha_r|\right)^k\prod_{r=1}^{j}\left(p_{0,r}^{-|\alpha_r|}\right)\right)\leqslant\sum_{\alpha\in\mathbb{Z}^j}\left(\left(\sum_{r=1}^{j}|\alpha_r|\right)^k(\sqrt{p_{0,j}})^{-\sum_{r=1}^{j}|\alpha_r|}\prod_{r=1}^{j}\left(\frac{1}{\sqrt{p_{0,r}}}\right)^{|\alpha_r|}\right)$$

$$\leqslant\frac{\tilde{C}_k}{(\ln(p_{0,j}))^k}\sum_{\alpha\in\mathbb{Z}^j}\left(\prod_{r=1}^{j}\left(\frac{1}{\sqrt{p_{0,r}}}\right)^{|\alpha_r|}\right)=\frac{\tilde{C}_k}{(\ln(p_{0,j}))^k}\prod_{r=1}^{j}\left(1+\frac{2}{\sqrt{p_{0,r}}-1}\right).$$

Then (13) follows from (19) and from the last inequality with $\tilde{P}_k = C_{2,k}^k\tilde{C}_k$. □

Remark. In the case $k = 0$ inequality (13) holds with $\tilde{P}_0 = 1$.

4. Estimates on the EFGP angles

In the following sections we will need some estimates on the derivatives of the EFGP angles $\bar{\theta}_n$. The corresponding estimates in the case of a bounded potential are contained in [4, 3] (in the last paper only the estimates for the first two derivatives are proved). In the case under consideration we prove the following lemma:

Lemma 4.1. *Let v_n and x_n obey (2) and (3). Then there exist constants C_l, $l = 0, 1, 2, \ldots$, such that the following inequalities hold:*

$$\sup_{x\in(a,b)}|\bar{\theta}^{(l)}(x_n+1)|\leqslant C_l x_n^l\left(v_n^2+4\right)^{2l}\qquad n,l=1,2,\ldots$$

$$\sup_{x\in(a,b)}\left|\frac{\mathrm{d}\bar{\theta}(x_n)}{\mathrm{d}x}-x_n\right|\leqslant C_0x_{n-1}v_{n-1}^4\qquad n=2,3,\ldots.$$

Proof. Because of $\bar{\theta}'(x_n) = \theta'(x_n) + 1$ we need only to prove the above estimates with $\bar{\theta}(x_n)$ replaced by $\theta(x_n)$. We use the simple inequality

$$\min_{\varphi\in[0,2\pi]}(1-v\sin 2\varphi+v^2\sin^2\varphi)\geqslant(v^2+4)^{-1}\tag{20}$$

which can be proved by the methods of elementary calculus. We differentiate equation (7) and solve for θ_n' to obtain

$$\begin{aligned}\theta'(n+1)=&\frac{\theta'(n)+1}{1-\frac{V(n)}{\sin x}\sin(2\bar{\theta}(n))+\frac{V(n)^2}{\sin^2 x}\sin^2(\bar{\theta}(n))}\\&-\frac{\cos x\sin^2(\bar{\theta}(n))V(n)}{\sin^2 x\left(1-\frac{V(n)}{\sin x}\sin(2\bar{\theta}(n))+\frac{V(n)^2}{\sin^2 x}\sin^2(\bar{\theta}(n))\right)}.\end{aligned}\tag{21}$$

From this and (2) follows

$$\theta'(x_{n+1})=\theta'(x_n+1)+x_{n+1}-x_n-1\qquad\theta^{(l)}(x_{n+1})=\theta^{(l)}(x_n+1)\quad l\geqslant 2.\tag{22}$$

Using (20) we then obtain from (21) the inequality

$$|\theta'(x_n+1)|\leqslant|\theta_n'+1|\left(\frac{v_n^2}{a_0^2}+4\right)+\frac{v_n\left(\frac{v_n^2}{a_0^2}+4\right)}{a_0^2}.$$

We can continue the last inequality using (22) as follows:

$$|\theta'(x_n+1)| \leqslant \frac{1}{a_0^2}(|\theta'(x_{n-1}+1)| + x_n - x_{n-1})\left(v_n^2+4\right) + \frac{v_n\left(v_n^2+4\right)}{a_0^4}$$
$$\leqslant \frac{x_n}{a_0^2}\left(v_n^2+4\right)^2\left(\frac{|\theta'(x_{n-1}+1)|}{x_n\left(v_n^2+4\right)} + \frac{1}{\left(v_n^2+4\right)} + \frac{1}{x_n a_0^2}\right).$$

We can now choose a constant C_1 sufficiently large, so that by induction from the last estimate (with the help of (2) and (3)) follows for all n the inequality

$$\sup_{x\in(a,b)} |\theta'(x_n+1)| \leqslant C_1 x_n\left(v_n^2+4\right)^2. \tag{23}$$

From (22) and (23) follows then

$$|\theta'(x_n) - x_n| \leqslant |\theta'(x_{n-1})| + x_{n-1} \leqslant C_1 x_{n-1}\left(v_{n-1}^2+4\right)^2 + x_{n-1} \leqslant 2C_1 x_{n-1} v_{n-1}^4$$

where we probably have to enlarge C_1. Then the claim of the present lemma for the first derivatives follows with $C_0 = 2C_1$.

To prove the assertion for the higher derivatives, we have to differentiate (21) sufficiently many times. As a result, we obtain for $l \geqslant 2$ the formula

$$\theta^{(l)}(n+1) = \frac{\theta^{(l)}(n)}{1 - \frac{V(n)}{\sin x}\sin(2\bar\theta(n)) + \frac{V(n)^2}{\sin^2 x}\sin^2(\bar\theta(n))} + \frac{P_l(n)}{\sin^{4l} x\left(1 - \frac{V(n)}{\sin x}\sin(2\bar\theta(n)) + \frac{V(n)^2}{\sin^2 x}\sin^2(\bar\theta(n))\right)^l} \tag{24}$$

where $P_l(n)$ is a real polynomial of the form

$$P_l(n) = \sum_{\beta\in J} c_\beta(\sin x)^{a_\beta}(\cos x)^{b_\beta}(\sin(\bar\theta(n)))^{s_\beta}(\cos(\bar\theta(n)))^{r_\beta}V(n)^{w_\beta}\prod_{j=1}^{l-1}(\theta^{(j)}(n))^{u_{j,\beta}}$$

for which holds: J is a finite set of indices, $w_\beta \leqslant 2l-1$, $0 \leqslant u_{j,\beta}$ and $\sum_{j=1}^{l-1} j u_{j,\beta} \leqslant l$ for all β from J. (This is easy to show by induction on l.) We can estimate $P_l(n)$ as follows:

$$|P_l(n)| \lesssim V(n)^{2l-1} \sup_{(u_1,\dots,u_{l-1}):\sum_{j=1}^{l-1} j u_j \leqslant l} \prod_{j=1}^{l-1}|(\theta^{(j)}(n))^{u_j}|.$$

From formula (24) we obtain the inequality

$$|\theta^{(l)}(x_n+1)| \leqslant |\theta^{(l)}(x_n)|\left(\frac{v_n^2}{a_0^2}+4\right) + \frac{|P_l(x_n)|\left(\frac{v_n^2}{a_0^2}+4\right)^l}{a_0^{4l}}. \tag{25}$$

We will now use the induction on l. So let us assume that

$$\sup_{x\in[a,b]} |\theta^{(j)}(x_n+1)| \leqslant C_j x_n^j\left(v_n^2+4\right)^{2j}$$

for each $j \leqslant l-1$. Then we can modify the estimate for P_l as follows:

$$\sup_{x\in[a,b]} |P_l(x_n)| \lesssim v_n^{2l-1} \sup_{\sum_{j=1}^{l-1} j u_j \leqslant l} \left(x_n + C_0 v_{n-1}^4 x_{n-1}\right)^{u_1}\left(\prod_{j=2}^{l-1} C_j\right)\left(v_{n-1}^4 x_{n-1}\right)^{\sum_{j=2}^{l-1} j u_j} \lesssim v_n^{2l-1} x_n^l$$

(here we have used (22), the definition of P_l and (3)).

Thus we can obtain from (25) using the last estimate on P_l and (22) the estimate

$$|\theta^{(l)}(x_n+1)| \leqslant |\theta^{(l)}(x_{n-1}+1)| \left(\frac{v_n^2}{a_0^2}+4\right) + Cx_n^l v_n^{2l-1} \frac{\left(\frac{v_n^2}{a_0^2}+4\right)^l}{a_0^{4l}}$$
$$= \frac{x_n^l\left(v_n^2+4\right)^{2l}}{a_0^2}\left(\frac{|\theta^{(l)}(x_{n-1}+1)|}{\left(v_n^2+4\right)^{2l-1}x_n^l} + \frac{C}{a_0^{6l-2}}\frac{v_n^{2l-1}}{\left(v_n^2+4\right)^l}\right).$$

We can now choose a constant C_l sufficiently large, so that by induction on n from the last estimate follows for all n the inequality

$$\sup_{x\in(a,b)} |\theta'(x_n+1)| \leqslant C_l x_n^l \left(v_n^2+4\right)^{2l}. \tag{26}$$

(We have used here again condition (3).)

Thus the induction on l is also complete and the estimates on the higher derivatives are proved. □

Remark. We can weaken the conditions of the previous lemma. Actually, it suffices to assume the relations

$$\lim_{n\to\infty}\frac{1}{x_n} = \lim_{n\to\infty}\frac{1}{v_n} = \lim_{n\to\infty}\frac{x_n v_n^2}{x_{n+1}} = 0. \tag{27}$$

Corollary 4.2. *Under the same assumptions as in the previous lemma holds*

$$\lim_{j\to\infty}\frac{\bar\theta'(x_j)}{x_j} = 1.$$

Remark. Let $\varepsilon > 0$ be chosen. Then we can assume without loss of generality that $(1-\varepsilon)x_j < \bar\theta'_j < (1+\varepsilon)x_j$ holds not only for sufficiently large j (which is the claim of the last corollary), but also for all j. For this we have to take in theorem 2.2

$$R^{-2}(x_{j+j_0},k) = R^{-2}(x_{j_0},k)\prod_{r=j_0+1}^{j-1}\left(1-\frac{v_r\sin(2\bar\theta_r)}{\sin x}+\frac{v_r^2\sin^2(\bar\theta_r)}{\sin^2 x}\right)^{-1}$$

instead of (10), define h there by $h(x) = \frac{2}{\pi}R^{-2}(x_{j_0},x)f(2\cos x)\sin^2 x$ and then renumerate correspondingly x_j, v'_j and $\bar\theta'_j$.

Corollary 4.3. *Using (3) and (22) we can rewrite the statement of lemma 4.1 as follows:*

$$\sup_{x\in[a,b]} |\bar\theta^{(l)}(x_{n+1})| \leqslant C_l x_n^{\frac{(2-\delta)l}{\delta}} \qquad n=1,2,\dots \quad l=2,3,\dots$$
$$\sup_{x\in(a,b)}\left|\frac{\mathrm{d}\bar\theta(x_n)}{\mathrm{d}x}-x_n\right| \leqslant C_0 x_{n-1}^{\frac{2-\delta}{\delta}} \qquad n=2,3,\dots$$

where we probably have to enlarge C_l, $l\geqslant 2$.

5. Non-resonant terms

We now describe briefly our general strategy for estimating (9). We consider separate integrals from this sum, that is the integrals of the form

$$\int_a^b h(x)C_j^\alpha(x)\exp\left(\mathrm{i}\,2\left(-t\cos x+\sum_{r=1}^{j}(\alpha_r\bar\theta_r)\right)\right)\mathrm{d}x. \tag{28}$$

By corollary 4.2, the derivative of the phase is roughly equal to

$$2\sum_{r=1}^{j}\alpha_r x_r + 2t\sin x.$$

Because of condition (3) (which implies the rapid growth of x_j) the last expression is, in most cases, of the order of $2\alpha_j x_j + 2t\sin x$. So, we can conclude that if $|t|$ is either much larger or much smaller than x_j and if the α_r with $r < j$ are not too large in the absolute value (otherwise $\left|\sum_{r=1}^{j}\alpha_r x_r\right|$ could be in some cases much smaller than $|\alpha_j|x_j$), the exponential oscillates heavily and the corresponding contribution to (9) is small in the absolute value (we call those terms 'non-resonant', exact definition will follow). So we have to cut off the series over the α_r. If $|t|$ is of the order of x_j, a different treatment is necessary (it occurs in the next section). In this case we will also cut off the series over the α_r.

For the exact determination of this cutting off we introduce the values γ_r^j:

$$\gamma_r^j = \left\{\begin{array}{ll} 4^{-1}x_j x_{j-1}^{\frac{\delta-2}{\delta}} & j > r \quad j > m(t)+1 \\ |t|^{\upsilon_0} & r = j = m(t)+1 \\ \left(2^{-1}|t|x_{j-1}^{\frac{\delta-2}{\delta}}\right)^{\rho_0} & r \leqslant m(t) \quad j = m(t)+1 \\ 4^{-1}|t|a_0 x_j^{\frac{\delta-2}{\delta}} & r \leqslant m(t) \quad j = m(t). \end{array}\right\}$$

The values ρ_0 and υ_0 are not defined for the moment, but we assume $0 < \rho_0 < 1$ and $0 < \upsilon_0 < 1$.

We use the notation 'non-resonant' for the terms (28) of three types:

(a) for the terms with $j > m(t)+1$ and with $|\alpha_r| \leqslant \gamma_r^j$ for all $r < j$,
(b) for the terms with $j = m(t)$ and with $|\alpha_r| \leqslant \gamma_r^{m(t)}$ for all $r \leqslant m(t)$,
(c) for the terms and with $j = m(t)+1$ and with $|\alpha_r| \leqslant \gamma_r^{m(t)+1}$ for all $r \leqslant m(t)+1$, for which holds

$$\inf_{x\in(a,b)} |t\sin(x) + \sum_{r=1}^{j}\alpha_r x_r| \geqslant \max\left(\frac{|t|a_0}{2}, \frac{|\alpha_j|x_j}{2}\right). \tag{29}$$

We denote the sets of corresponding α in all three cases with $A_{1,j}$.

For the terms (28) with $j = m(t)+1$ and with $|\alpha_r| \leqslant \gamma_r^{m(t)+1}$ for all $r \leqslant m(t)+1$, for which (29) does not hold, we use the notation 'resonant'. We denote the set of corresponding α with $A_{2,m(t)+1}$.

It is easy to see that we have not yet considered all α. The rest terms, which are not contained in non-resonant or resonant terms, have the following property: $\left|\alpha_{r_0}\right| > \gamma_{r_0}^j$ for some r_0. Therefore we refer to these terms also as terms 'with large max $|\alpha_r|$'. We denote the sets of corresponding α with $A_{3,j}$. (We note that the definition of the sets $A_{1,j}$, $A_{2,j}$ and $A_{3,j}$ depends, in the cases $j = m(t)$ and $j = m(t)+1$, on the value of t.)

We devote the rest of this section to the study of non-resonant terms. The discussion of the resonant terms follows in the next section and the consideration of rest terms occurs in section 7.

Let us start with the non-resonant case (a), so we consider terms (28) with $j > m(t)+1$ and $\alpha_r \leqslant \gamma_r^j$ for $r = 1, \ldots, j-1$. Abbreviating we obtain

$$K_j^\alpha = 2\left(-t\cos x + \sum_{r=1}^{j}\alpha_r\bar{\theta}_r\right).$$

Using corollary 4.2, we then see that if $\varepsilon > 0$ is chosen sufficiently small, for sufficiently large $|t|$ (note that this also ensures that j is large) holds

$$\inf_{x\in(a,b)} \left|\left(K_j^\alpha\right)'\right| \geqslant |\alpha_j|(1-\varepsilon)x_j - |t| - (1+\varepsilon)\sum_{r=1}^{j-1}|\alpha_r|x_r$$

$$\geqslant (1-\varepsilon)|\alpha_j|x_j - \frac{x_j}{2} - \frac{1+\varepsilon}{4}\frac{x_j}{x_{j-1}}\sum_{r=1}^{j-1}x_r \geqslant \frac{|\alpha_j|x_j}{4}. \tag{30}$$

In order to obtain good estimates, we must now integrate by parts sufficiently many times. To do this, we introduce the differential expression

$$L = \frac{-\mathrm{i}}{\left(K_j^\alpha\right)'}\frac{\mathrm{d}}{\mathrm{d}x}.$$

Note that $L\left(\mathrm{e}^{\mathrm{i}K_j^\alpha}\right) = \mathrm{e}^{\mathrm{i}K_j^\alpha}$. Therefore, we can manipulate integrals (28) as follows:

$$\int_a^b hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x = \int_a^b hC_j^\alpha\left(L^m\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\right)\mathrm{d}x = \int_a^b \mathrm{e}^{\mathrm{i}K_j^\alpha}\left[(L^*)^m\left(hC_j^\alpha\right)\right]\mathrm{d}x.$$

Here, $m \in \mathbb{N}$ is still to be chosen and

$$L^* = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{i}}{\left(K_j^\alpha\right)'(x)}$$

is the transpose of L. There are no boundary terms because the support of h lies in $[a, b]$. Thus we obtain the estimate

$$\left|\int_a^b h(x)C_j^\alpha(x)\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \leqslant \pi \max_{x\in(a,b)}\left|(L^*)^m\left(h(x)C_j^\alpha(x)\right)\right|. \tag{31}$$

So, our next task is to control $(L^*)^m\left(hC_j^\alpha\right)$.

Lemma 5.1. *For each positive integer m there exists a constant $\tilde{B}_m$, such that holds the inequality*

$$\sup_{x\in(a,b)}\left|(L^*)^m\left(h(x)C_j^\alpha(x)\right)\right| \leqslant \tilde{B}_m \sup_{\substack{m\leqslant l\leqslant 2m\\ x\in(a,b)}}\ \sup_{\substack{(\kappa,\zeta_1,\ldots,\zeta_{l-1})\in(\mathbb{Z}_+)^l:\\ \kappa\leqslant m,\\ \sum_{k=1}^{l-1}(k+1)\zeta_k\leqslant l-\kappa}}$$

$$\times\left|\left(C_j^\alpha(x)\right)^{(\kappa)}\frac{\prod_{k=1}^{l-1}\left(\left(K_j^\alpha(x)\right)^{(k+1)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha(x)\right)'\right)^l}\right|.$$

Proof. For each m there exist constants $C_{(\kappa,\zeta)}^{m,l}$, independent of j, x and α (some of them can be equal to zero), such that

$$(L^*)^m\left(hC_j^\alpha\right) = \sum_{l=m}^{2m}\sum_{(\kappa,\zeta)\in I_l}\frac{C_{(\kappa,\zeta)}^{m,l}\left(hC_j^\alpha\right)^{(\kappa)}\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(k+1)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^l} \tag{32}$$

where I_l is defined by $I_l = \{(\kappa,\zeta)\in\{0,\ldots,l\}\times\mathbb{Z}_+^{l-1} : \kappa + \sum_{k=1}^{l-1}(k+1)\zeta_k \leqslant l\}$. Formula (32) is, in the case $m = 1$, directly obtained by derivation. (We have in this case

$$I_1 = \{(0),(1)\} \qquad I_2 = \{(0,0),(1,0),(2,0),(0,1)\}$$

$C_{(1)}^{1,1} = \mathrm{i} = -C_{(0,1)}^{1,2}$ and else $C_{(\kappa,\zeta)}^{1,l}$ are equal to zero.)

For arbitrary $m > 1$, (32) is easily proved by induction argument, using

$$L^*\left(\frac{\left(hC_j^\alpha\right)^{(\kappa)}\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(k+1)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^l}\right)=\frac{\mathrm{i}\left(hC_j^\alpha\right)^{(\kappa+1)}\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(k+1)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^{l+1}}$$
$$+\mathrm{i}\left(hC_j^\alpha\right)^{(\kappa)}\sum_{k_0=1}^{l-1}\frac{\zeta_{k_0}\left(\left(K_j^\alpha\right)^{(k_0+1)}\right)^{\zeta_{k_0}-1}\left(K_j^\alpha\right)^{(k_0+2)}\prod_{k\neq k_0}\left(\left(K_j^\alpha\right)^{(k+1)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^{l+1}}$$
$$-\frac{\mathrm{i}(l+1)\left(hC_j^\alpha\right)^{(\kappa)}\left(K_j^\alpha\right)''\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(k+1)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^{l+2}}.$$

The sets I_l are evidently finite and, moreover, $|I_l|$ (the cardinality of I_l) depends only on l. Then, taking in account that h lies in $C_0^\infty(-2, 2)$, the present lemma follows from (32). □

To bound the expressions $\prod_{k=1}^{l-1}\left(\left(K_j^\alpha(x)\right)^{(k+1)}\right)^{\zeta_k}$, we use corollary 4.3 which implies that

$$\left|\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(k+1)}\right)^{\zeta_k}\right|\leqslant\prod_{k=1}^{l-1}\left(|t|+\sum_{r=1}^{j}|\alpha_r|C_{\xi_k}x_{r-1}^{\frac{(k+1)(2-\delta)}{\delta}}\right)^{\zeta_k}. \tag{33}$$

From the last inequality and inequality (30) follows the estimate

$$\left|\frac{\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(k+1)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^l}\right|\leqslant\frac{4^l}{(|\alpha_j|x_j)^\nu}\prod_{k=1}^{l-1}\left(\frac{|t|+\sum_{r=1}^j|\alpha_r|C_{k+1}x_{r-1}^{\frac{(k+1)(2-\delta)}{\delta}}}{(|\alpha_j|x_j)^{k+1}}\right)^{\zeta_k} \tag{34}$$

with $\nu = l - \sum_{k=1}^{l-1}(k+1)\zeta_k$. We have now to consider the separate factors from this product. So let ξ be any integer $\geqslant 2$. Then we use $|t| \leqslant x_j$ (which follows from $j > m(t) + 1$) and (3) to obtain the estimates

$$\frac{|t|+\sum_{r=1}^j|\alpha_r|C_\xi x_{r-1}^{\frac{\xi(2-\delta)}{\delta}}}{(|\alpha_j|x_j)^\xi}\leqslant\frac{|t|+x_jx_{j-1}^{\frac{\delta-2}{\delta}}\sum_{r=1}^{j-1}\left(C_\xi x_{r-1}^{\frac{\xi(2-\delta)}{\delta}}\right)+|\alpha_j|C_\xi x_{j-1}^{\frac{\xi(2-\delta)}{\delta}}}{(|\alpha_j|x_j)^\xi}$$
$$\leqslant\frac{|t|}{x_j^\xi}+C_\xi x_{j-1}^{\frac{(\delta-2)}{\delta}}x_j^{1-\xi}\sum_{r=1}^{j-1}x_{r-1}^{\frac{\xi(2-\delta)}{\delta}}+C_\xi\frac{x_{j-1}^{\frac{\xi(2-\delta)}{\delta}}}{x_j^\xi}\lesssim\frac{x_{j-1}^{\frac{(\xi-1)(2-\delta)}{\delta}}}{x_j^{\xi-1}}.$$

From the last estimate and (34) it follows that there exists a constant G_l such that

$$\sup_{x\in(a,b)}\left|\frac{\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(k+1)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^l}\right|\leqslant\frac{G_l}{x_j^\nu}\prod_{k=1}^{l-1}\left(\frac{x_{j-1}^{\frac{k(2-\delta)}{\delta}}}{x_j^k}\right)^{\zeta_k}=\frac{G_l}{x_j^\nu}\left(\frac{x_{j-1}^{\frac{2-\delta}{\delta}}}{x_j}\right)^{\sum_{k=1}^{l-1}k\zeta_k}.$$

We continue this with the help of the estimates

$$\nu+\sum_{k=1}^{l-1}k\zeta_k=l-\sum_{k=1}^{l-1}\zeta_k\geqslant l-\frac{1}{2}\sum_{k=1}^{l-1}(k+1)\zeta_k\geqslant\frac{l+\kappa}{2}$$

to obtain the inequality

$$\sup_{\substack{x\in(a,b)\\ \sum_{k=1}^{l-1}(k+1)\zeta_k\leqslant l-\kappa}}\left|\frac{\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(k+1)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^l}\right|\leqslant G_l\frac{x_{j-1}^{\frac{(2-\delta)l}{\delta}}}{x_j^{\frac{l+\kappa}{2}}}. \tag{35}$$

Proposition 5.2. *Let the sequences (v_n) and (x_n) obey (3). Let δ be arbitrary from $\left(\frac{1}{2},1\right)$. Then there exist for any positive integer n a constant B_n and t_0 from $\mathbb{R}$, such that for $|t|>t_0$ and $j>m(t)+1$ holds the inequality*

$$\sum_{\alpha\in A_{1,j}}\left|\int_a^b h(x)C_j^\alpha(x)\,\mathrm{e}^{\mathrm{i}K_j^\alpha(x)}\,\mathrm{d}x\right|\leqslant B_n x_j^{-n}.$$

Proof. Let n be fixed. Using (31), (35) and lemma 5.1, we obtain the estimate

$$\left|\int_a^b h(x)C_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right|\leqslant \tilde{G}_m \sup_{\substack{0\leqslant\kappa\leqslant m\\ m\leqslant l\leqslant 2m\\ x\in(a,b)}}\left|\left(C_j^\alpha(x)\right)^{(\kappa)}\right|\frac{x_{j-1}^{\frac{(2-\delta)l}{\delta}}}{x_j^{\frac{l+\kappa}{2}}} \tag{36}$$

with $\tilde{G}_m=\pi\tilde{B}_m\max_{m\leqslant l\leqslant 2m}G_l$. We have now to use lemma 3.1. By Taylor expansion we obtain

$$\sqrt{p_{0,r}}=\sqrt[4]{\frac{v_r^2+4a_0^2}{v_r^2}}=\sqrt[4]{1+\frac{4a_0^2}{v_r^2}}=1+\frac{a_0^2}{v_r^2}+o\left(\frac{1}{v_r^3}\right)$$
$$\ln(p_{0,r})=\ln\left(\sqrt{1+\frac{4a_0^2}{v_r^2}}\right)=\frac{2a_0^2}{v_r^2}+o\left(\frac{1}{v_r^3}\right).$$

From these expansions, from lemma 3.1 (formula (13)) and from condition (3) follows for each $\varepsilon>0$:

$$\sum_{\alpha\in\mathbb{Z}^j}\sup_{x\in(a,b)}\left|\left(C_j^\alpha(x)\right)^{(\kappa)}\right|\lesssim v_j^{2\kappa+2+2\varepsilon}=x_j^{\frac{(1-\delta)(\kappa+1+\varepsilon)}{\delta}}. \tag{37}$$

From this estimate and from (36) we conclude that there exists for each m a constant K_m, such that

$$\sum_{\alpha\in A_{1,j}}\left|\int_a^b h(x)C_j^\alpha(x)\,\mathrm{e}^{2\mathrm{i}K_j^\alpha(x)}\,\mathrm{d}x\right|\leqslant K_m\sup_{\substack{0\leqslant\kappa\leqslant m\\ m\leqslant l\leqslant 2m}}\frac{x_{j-1}^{\frac{(2-\delta)l}{2\delta}}}{x_j^{\frac{l+\kappa}{2}-\frac{(1-\delta)(\kappa+1+\varepsilon)}{\delta}}}. \tag{38}$$

From $\delta>\frac{1}{2}$ follows $\frac{(1-\delta)}{\delta}<1-\tilde{\varepsilon}$ with some $\tilde{\varepsilon}>0$ and we can choose m so that holds $\left(1+2\frac{1+\varepsilon}{m}\right)\frac{1-\delta}{\delta}<1-\tilde{\varepsilon}$. Then we have for all $\kappa\geqslant\frac{l}{2}\ \left(\geqslant\frac{m}{2}\right)$ the relation

$$\frac{l+\kappa}{2}-\frac{(1-\delta)(\kappa+1+\varepsilon)}{\delta}=\frac{l}{2}+\frac{\kappa}{2}\left(1-\left(1+\frac{1+\varepsilon}{\kappa}\right)\frac{2(1-\delta)}{\delta}\right)>\frac{l+2\kappa\tilde{\varepsilon}-\kappa}{2}\geqslant\kappa\tilde{\epsilon}.$$

For all κ with $\kappa<\frac{l}{2}$ holds

$$\frac{l+\kappa}{2}-\frac{(1-\delta)(\kappa+1+\varepsilon)}{\delta}\geqslant\frac{l+\kappa}{2}-\kappa-1-\varepsilon=\frac{l-\kappa}{2}-1-\varepsilon>\frac{l}{4}-1-\varepsilon.$$

Thus we can continue (38) as follows:

$$\sum_{\alpha\in A_{1,j}}\left|\int_a^b h(x)C_j^\alpha(x)\,\mathrm{e}^{2\mathrm{i}K_j^\alpha(x)}\,\mathrm{d}x\right|\leqslant K_m\frac{x_{j-1}^{\frac{(2-\delta)m}{\delta}}}{x_j^{\min\left\{\frac{m\tilde{\varepsilon}}{2},\frac{m}{4}-1-\varepsilon\right\}}}.$$

We have now only to use condition (3) to obtain for large j the inequality $x_{j-1}^{\frac{(2-\delta)m}{\delta}}\leqslant x_j$. Then the present proposition follows with sufficiently large t_0 (which ensures that all j under consideration are also large) and with $B_n=K_m$, where m is chosen so that holds $\min\left\{\frac{m\tilde{\varepsilon}}{2},\frac{m}{4}-1-\varepsilon\right\}>n+1$. □

The non-resonant cases (b) and (c) can be treated similarly. We will thus keep in these cases the discussion brief. First we consider case (b). It holds $j = m(t)$ and $\alpha_r \leqslant \gamma_r^j$ for $r = 1, \dots, j$. Instead of (30) we have with sufficiently small $\varepsilon > 0$

$$\inf_{x\in(a,b)} \left|\left(K_j^\alpha\right)'\right| \geqslant |t| \inf_{x\in(a,b)} \sin(x) - (1+\varepsilon)\sum_{r=1}^{j} |\alpha_r| x_r \geqslant |t|a_0 - (1+\varepsilon)\sum_{r=1}^{j} \frac{|t|a_0}{4x_j} x_r$$
$$= |t|a_0 - \frac{(1+\varepsilon)|t|a_0}{4}\left(1 + x_j^{-1}\sum_{r=1}^{j-1} x_r\right) \geqslant \frac{|t|a_0}{2}. \tag{39}$$

Lemma 5.1 holds also in this case without any change. We can also use inequality (33), which is also valid. Instead of (34) we have

$$\sup_{x\in(a,b)} \left| \frac{\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(\xi_k)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^l} \right| \leqslant \frac{2^l}{(|t|a_0)^\nu} \prod_{k=1}^{l-1} \left(\frac{|t| + \sum_{r=1}^j |\alpha_r| C_{\xi_k} x_{r-1}^{\frac{(k+1)(2-\delta)}{\delta}}}{(|t|a_0)^{k+1}} \right)^{\zeta_k}. \tag{40}$$

So we need to consider the separate factors from this product. We obtain for any $\xi \geqslant 2$ the following sequence of estimates:

$$\frac{|t| + \sum_{r=1}^j |\alpha_r| C_{\xi_k} x_{r-1}^{\frac{\xi(2-\delta)}{\delta}}}{(|t|a_0)^\xi} \leqslant \frac{|t| + \sum_{r=1}^j |t| a_0 x_j^{\frac{\delta-2}{\delta}} C_\xi x_{r-1}^{\frac{\xi(2-\delta)}{\delta}}}{|t|^\xi}$$
$$\leqslant |t|^{1-\xi} + C_\xi a_0 \left(\frac{x_j^{\frac{2-\delta}{\delta}}}{|t|}\right)^{\xi-1} \left(\frac{1}{x_j}\right)^{\xi\frac{2-\delta}{\delta}} \sum_{r=1}^{j} x_{r-1}^{\frac{\xi(2-\delta)}{\delta}} \lesssim \left(\frac{x_j^{\frac{2-\delta}{\delta}}}{|t|}\right)^{\xi-1}. \tag{41}$$

Thus we obtain instead of (35) the following inequality:

$$\sup_{\substack{x\in(a,b)\\ \sum_{k=1}^l (k+1)\zeta_k \leqslant l-\kappa}} \left| \frac{\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(k+1)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^l} \right| \leqslant G_l \frac{x_j^{\frac{(2-\delta)l}{\delta}}}{|t|^{\frac{l+\kappa}{2}}} \leqslant G_l \frac{x_j^{\frac{(2-\delta)l}{\delta}}}{|t|^{\frac{l}{2}}}. \tag{42}$$

(These constants G_l are possibly different from G_l of (35).)

Proposition 5.3. *Let the sequences* (v_n) *and* (x_n) *obey (3). Let* δ *be arbitrary from* $(0, 1)$. *Then there exist for any positive integer n a constant* B_n *and* $t_0 > 0$, *so that for* $|t| > t_0$ *and for* $j = m(t)$ *holds the inequality*

$$\sum_{\alpha\in A_{1,j}} \left| \int_a^b h(x) C_j^\alpha(x)\, \mathrm{e}^{\mathrm{i}K_j^\alpha(x)}\, \mathrm{d}x \right| \leqslant B_n |t|^{-n}.$$

Proof. Let n be fixed. Using (31), (42) and lemma 5.1 we obtain in this case the following estimate (compare with the proof of proposition 5.2):

$$\left| \int_a^b h(x) C_j^\alpha\, \mathrm{e}^{\mathrm{i}K_j^\alpha}\, \mathrm{d}x \right| \lesssim \sup_{\substack{0\leqslant\kappa\leqslant m\\ x\in(a,b)}} \left| \left(C_j^\alpha(x)\right)^{(\kappa)} \right| \left(\frac{x_j^{\frac{2(2-\delta)}{\delta}}}{|t|} \right)^{\frac{m}{2}}. \tag{43}$$

Estimate (37) from the proof of proposition 5.2 remains valid in this case also. From this estimate and from (43) we conclude that there exists for each m a constant K_m, so that holds

$$\sum_{\alpha\in A_{1,j}} \left| \int_a^b h(x) C_j^\alpha(x)\, \mathrm{e}^{2\mathrm{i}K_j^\alpha(x)}\, \mathrm{d}x \right| \leqslant K_m \frac{x_j^{\frac{(2-\delta)m+(1-\delta)(m+1+\varepsilon)}{\delta}}}{|t|^{\frac{m}{2}}}.$$

Then the present proposition follows from (3) with $B_n = K_{2n+1}$ and t_0 sufficiently large. □

Now we come to case (c). We denote $\min_{x\in[a,b]} \left|t\sin(x) + \sum_{r=1}^{j}\alpha_r x_r\right|$ with D_t^α. Let x_0 be the point where this minimum is achieved. Then we have $D_t^\alpha = \left|t\sin(x_0) + \sum_{r=1}^{j}\alpha_r x_r\right|$. We use (3), (29) and corollary 4.3 to obtain for large $j(= m(t)+1)$ the following sequence of inequalities:

$$\inf_{x\in(a,b)} \left|\left(K_j^\alpha(x)\right)'\right| \geqslant \left|t\sin(x_0) + \sum_{r=1}^{j}\alpha_r x_r\right| - \sum_{r=1}^{j}|\alpha_r||\bar{\theta}_r' - x_r|$$

$$\geqslant \left|t\sin(x_0) + \sum_{r=1}^{j}\alpha_r x_r\right| - C_0|\alpha_j| x_{j-1}^{\frac{2-\delta}{\delta}} - \sum_{r=1}^{j-1}\left(x_j x_{j-1}^{\frac{\delta-2}{\delta}}\right)^{\rho_0} C_0 x_{r-1}^{\frac{2-\delta}{\delta}}$$

$$\geqslant D_t^\alpha\left(1 - 2C_0\left(\frac{x_{j-1}^{\frac{2-\delta}{\delta}}}{x_j} + \left(\frac{x_{j-1}^{\frac{2-\delta}{\delta}}}{x_j}\right)^{1-\rho_0}\frac{\sum_{r=1}^{j-1}\left(x_{r-1}^{\frac{2-\delta}{\delta}}\right)}{x_{j-1}^{\frac{2-\delta}{\delta}}}\right)\right) \geqslant \frac{D_t^\alpha}{2} \geqslant \frac{|\alpha_j|x_j}{4}.$$

So, instead of (34) from the case (a) and (40) from the case (b) we have in the case (c) the estimate

$$\sup_{x\in(a,b)}\left|\frac{\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(\xi_k)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^l}\right| \leqslant \frac{2^l}{(D_t^\alpha)^\nu}\prod_{k=1}^{l-1}\left(\frac{|t| + \sum_{r=1}^{j}|\alpha_r| C_{\xi_k} x_{r-1}^{\frac{(k+1)(2-\delta)}{\delta}}}{(D_t^\alpha)^{k+1}}\right)^{\zeta_k}. \tag{44}$$

We consider again the separate factors. We use the inequality

$$|t| = \frac{|t\sin x_0|}{\sin x_0} \leqslant \frac{D_t^\alpha + \sum_{r=1}^{j}|\alpha_r| x_r}{\sin x_0}$$

to obtain for each integer $\xi \geqslant 2$ the following estimates:

$$\frac{|t| + \sum_{r=1}^{j}|\alpha_r| C_\xi x_{r-1}^{\frac{\xi(2-\delta)}{\delta}}}{\left|t\sin(x_0) + \sum_{r=1}^{j}\alpha_r x_r\right|^\xi}$$

$$\leqslant \frac{D_t^\alpha + \sum_{r=1}^{j}|\alpha_r| x_r + \sin(x_0)\left(|\alpha_j| C_\xi x_{j-1}^{\frac{\xi(2-\delta)}{\delta}} + C_\xi \frac{x_j}{x_{j-1}}\sum_{r=1}^{j-1} x_{r-1}^{\frac{\xi(2-\delta)}{\delta}}\right)}{\sin(x_0)\,(D_t^\alpha)^\xi}$$

$$\lesssim \left(\frac{1}{D_t^\alpha}\right)^{\xi-1} + \frac{|\alpha_j|\left(x_{j-1}^{\frac{\xi(2-\delta)}{\delta}} + x_j\right)}{(D_t^\alpha)^{\xi-1}|\alpha_j| x_j} \lesssim \frac{x_{j-1}^{\frac{(\xi-1)(2-\delta)}{\delta}}}{(D_t^\alpha)^{\xi-1}} \lesssim \left(\frac{x_{j-1}^{\frac{(2-\delta)}{\delta}}}{t}\right)^{\xi-1}. \tag{45}$$

We obtain from this as in the previous case

$$\sup_{\substack{x\in(a,b)\\ \sum_{k=1}^{l}(k+1)\zeta_k\leqslant l}}\left|\frac{\prod_{k=1}^{l-1}\left(\left(K_j^\alpha\right)^{(k+1)}\right)^{\zeta_k}}{\left(\left(K_j^\alpha\right)'\right)^l}\right| \leqslant G_l\left(\frac{x_{j-1}^{\frac{2(2-\delta)}{\delta}}}{|t|}\right)^{\frac{l}{2}}. \tag{46}$$

(These constants G_l are possibly different from G_l of (35) and (42).)

Proposition 5.4. *Let the sequences* (v_n) *and* (x_n) *obey (3). Let* δ *be arbitrary from* $(0,1)$. *Then there exist for any positive integer n a constant* B_n *and* $t_0 > 0$, *so that for* $|t| > t_0$ *and for* $j = m(t)+1$ *holds the inequality*

$$\sum_{\alpha\in A_{1,j}}\left|\int_a^b h(x) C_j^\alpha(x)\,\mathrm{e}^{\mathrm{i}K_j^\alpha(x)}\,\mathrm{d}x\right| \leqslant B_n|t|^{-n}.$$

Proof. We have only to repeat the proof of proposition 5.3 with (42) replaced by (46). □

6. Resonant terms

We consider in this section the resonant terms, that is terms (28) with $j = m(t) + 1$, $\alpha_r \leqslant \gamma_r^j$ for $1 \leqslant r \leqslant j$, for which holds

$$\min_{x\in[a,b]} \left| t\sin(x) + \sum_{r=1}^{j} \alpha_r x_r \right| < \frac{\max\{|\alpha_j x_j|, |t|a_0\}}{2}. \tag{47}$$

For the sake of brevity, we write in this section j for $m(t)+1$. Abbreviate $g(x) = 2\big(t\sin(x) + \sum_{r=1}^{j} \alpha_r x_r\big)$. The point $x = \pi/2$ (which corresponds to the energy $E = 0$) plays a special role now because the derivative of g is zero there. Therefore, we assume first that $\pi/2 \notin [a, b]$ (the opposite case we will consider further).

The terms under consideration have possibly small $\inf_{x\in(a,b)} \big|\big(K_j^\alpha\big)'\big|$, so we cannot proceed in this case as in the previous section. To avoid this difficulty we represent K_j^α as follows: $K_j^\alpha = \eta_1 + \eta_2$, where

$$\eta_1 = 2\left(-t\cos(x) + x\sum_{r=1}^{j}\alpha_r x_r\right) \qquad \eta_2 = 2\sum_{r=1}^{j}\alpha_r(\bar\theta_r - x x_r).$$

Let ε_1 be any positive number. The function $g(x)$ (which is the derivative of η_1) is monotone on (a, b), therefore there exists only one point of minimum of $|g(x)|$. We denote this point with x_0 and interval $(x_0 - \varepsilon_1, x_0 + \varepsilon_1)\bigcap(a, b)$ with I_1. Then we have for all $x \notin I_1$

$$2\left|t\sin(x) + \sum_{r=1}^{j}\alpha_r x_r\right| \geqslant |t||\sin x - \sin x_0| \geqslant M|t|\varepsilon_1 \tag{48}$$

with $M = \inf(|\cos x|, x \in (a, b)) > 0$.

We split integral (28) and then integrate by parts to obtain

$$\begin{aligned}
\int_a^b hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x &= \int_{I_1} hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x + \int_{(a,b)\setminus I_1}\frac{hC_j^\alpha\,\mathrm{e}^{\mathrm{i}\eta_2}}{\mathrm{i}g}\,\mathrm{d}\mathrm{e}^{\mathrm{i}\eta_1}\\
&= \int_{I_1} hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x - \left(\frac{hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}}{\mathrm{i}g}\right)_{x_0-\varepsilon_1}^{x_0+\varepsilon_1} - \int_{(a,b)\setminus I_1}\frac{h'C_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}}{\mathrm{i}g}\,\mathrm{d}x\\
&\quad - \int_{(a,b)\setminus I_1}\frac{h\frac{\mathrm{d}}{\mathrm{d}x}\big(C_j^\alpha\big)\,\mathrm{e}^{\mathrm{i}K_j^\alpha}}{\mathrm{i}g}\,\mathrm{d}x + \int_{(a,b)\setminus I_1}\frac{2hC_j^\alpha(t\cos(x))\,\mathrm{e}^{\mathrm{i}K_j^\alpha}}{\mathrm{i}g^2}\,\mathrm{d}x\\
&\quad - \int_{(a,b)\setminus I_1}\frac{2h(x)C_j^\alpha(x)\sum_{r=1}^{j}(\alpha_r(\bar\theta_r'(x) - x_r))\,\mathrm{e}^{\mathrm{i}K_j^\alpha}}{g}\,\mathrm{d}x.
\end{aligned} \tag{49}$$

We leave the first integral for the moment and estimate other summands. The only summand, which we cannot estimate immediately, is the last integral. So we consider the expressions $\big|g^{-1}\sum_{r=1}^{j}\alpha_r(\bar\theta_r'(x) - x_r)\big|$. From corollary 4.3 we have the inequality

$$\left|\sum_{r=1}^{j}\alpha_r(\bar\theta_r'(x) - x_r)\right| \leqslant C_0\left(\left(x_j x_{j-1}^{\frac{\delta-2}{\delta}}\right)^{\rho_0}\sum_{r=1}^{j-1} x_{r-1}^{\frac{2-\delta}{\delta}} + |t|^{\upsilon_0} x_{j-1}^{\frac{2-\delta}{\delta}}\right).$$

Then with (48) and with the definition of $g(x)$ follows the estimate

$$\sup_{x\in(a,b)\setminus I_1}\left|\frac{\sum_{r=1}^{j}\alpha_r(\bar\theta_r'(x)-x_r)}{\mathrm{i}g}\right| \leqslant \frac{C_0\left(|t|^{\upsilon_0}x_{j-1}^{\frac{2-\delta}{\delta}}+x_j^{\rho_0}x_{j-1}^{\frac{(\delta-2)\rho_0}{\delta}}\sum_{r=1}^{j-1}x_{r-1}^{\frac{2-\delta}{\delta}}\right)}{M\varepsilon_1|t|}$$

$$\lesssim \frac{x_{j-1}^{\frac{2-\delta}{\delta}}}{M\varepsilon_1|t|^{1-\upsilon_0}}+\frac{1}{M\varepsilon_1|t|^{1-\rho_0}}.$$

(The 'hidden' constant in the last estimate is independent of M and ε_1.) So we use (49) to obtain the estimate

$$\left|\int_a^b hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \leqslant \left|\int_{I_1} hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| + \frac{C(b-a)}{M\varepsilon_1|t|}\sup_{\substack{x\,\in\,(a,b)\\ \delta=0,1}}\left|\left(C_j^\alpha\right)^{(\delta)}\right|$$
$$+\left(\frac{C}{M|t|\varepsilon_1}+\frac{C(b-a)}{M\varepsilon_1}\left(|t|^{-1}+\frac{1}{M\varepsilon_1|t|}+\frac{x_{j-1}^{\frac{2-\delta}{\delta}}}{|t|^{1-\upsilon_0}}+\frac{1}{|t|^{1-\rho_0}}\right)\right)\sup_{x\in(a,b)}\left|C_j^\alpha\right|.$$

Let $(\varepsilon_k)_{k=1}^\infty$ be any monotonically decreasing sequence of positive numbers. (The first term of this sequence, that is ε_1, was already introduced). We define now inductively the sequence of intervals by $I_k=(x_0-\varepsilon_k,x_0+\varepsilon_k)\bigcap I_{k-1}$. Formula (49) holds with I_1 replaced by I_k, (a,b) replaced by I_{k-1} and with the additional boundary term

$$\left(\frac{hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}}{\mathrm{i}g}\right)_{x_0-\varepsilon_{k-1}}^{x_0+\varepsilon_{k-1}}$$

on the right-hand side. We can then repeat the previous procedure to obtain the following estimate (the contribution corresponding to the additional boundary term is small in comparison with the other terms and can be therefore omitted):

$$\left|\int_{I_{k-1}} hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \leqslant \left|\int_{I_k} hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| + \frac{C\varepsilon_{k-1}}{M\varepsilon_k|t|}\sup_{\substack{x\,\in\,(a,b)\\ \delta=0,1}}\left|\left(C_j^\alpha\right)^{(\delta)}\right|$$
$$+\left(\frac{C}{M|t|\varepsilon_k}+\frac{C\varepsilon_{k-1}}{M\varepsilon_k}\left(|t|^{-1}+\frac{1}{M\varepsilon_k|t|}+\frac{x_{j-1}^{\frac{2-\delta}{\delta}}}{|t|^{1-\upsilon_0}}+\frac{1}{|t|^{1-\rho_0}}\right)\right)\sup_{x\in(a,b)}\left|C_j^\alpha\right|.$$

For fixed m (which we have to specify further) follows from the last two estimates the inequality

$$\left|\int_a^b hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \leqslant C\left(\varepsilon_m\sup_{x\in(a,b)}\left|C_j^\alpha\right|+\sum_{k=1}^{m-1}\frac{\varepsilon_{k-1}}{M|t|\varepsilon_k}\sup_{\substack{x\,\in\,(a,b)\\ \delta=0,1}}\left|\left(C_j^\alpha\right)^{(\delta)}\right|\right)$$
$$+\sum_{k=1}^{m-1}\frac{C}{M\varepsilon_k}\left(\frac{2M\varepsilon_k+\varepsilon_{k-1}}{M\varepsilon_k|t|}+\frac{\varepsilon_{k-1}x_{j-1}^{\frac{2-\delta}{\delta}}}{|t|^{1-\upsilon_0}}+\frac{\varepsilon_{k-1}}{|t|^{1-\rho_0}}\right)\sup_{x\in(a,b)}\left|C_j^\alpha\right|. \qquad (50)$$

(We note again that C is independent of M and ε_k.)

Proposition 6.1. *Suppose $\pi/2\notin[a,b]$. Then for each $\sigma>0$ there exist $t_0>0$ and a constant C, so that for $|t|>t_0$ holds*

$$\sum_{\alpha\in A_{2,m(t)+1}}\left|\int_a^b hC_{m(t)+1}^\alpha\,\mathrm{e}^{\mathrm{i}K_{m(t)+1}^\alpha}\,\mathrm{d}x\right| \leqslant C|t|^{-\min\{\frac12,\frac{2\delta-1}{\delta},1-\upsilon_0,1-\rho_0\}+\frac{(1-\delta)}{\delta}+\sigma}.$$

Proof. We set $\varepsilon_k = |t|^{-k\mu}$ with $\mu > 0$ to obtain from (50), using (37), for sufficiently large $|t|$ the following inequality:

$$\sum_{\alpha\in A_{2,j}} \left|\int_a^b hC_j^\alpha \, \mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \leqslant C\left(|t|^{-m\mu} x_j^{\frac{(1-\delta)(1+\varepsilon)}{\delta}} + \frac{m}{M|t|^{1-\mu}} x_j^{\frac{(1-\delta)(2+\varepsilon)}{\delta}}\right)$$
$$+ C\left(\sum_{k=1}^{m-1}\left(\frac{2}{M^2|t|^{1-(k+1)\mu}}\right) + \frac{m x_{j-1}^{\frac{2-\delta}{\delta}}}{M|t|^{1-\upsilon_0-\mu}} + \frac{m|t|^{\rho_0}}{M|t|^{1-\mu}}\right) x_j^{\frac{(1-\delta)(1+\varepsilon)}{\delta}}.$$

We can continue the last estimate as follows:

$$\sum_{\alpha\in A_{2,j}} \left|\int_a^b hC_j^\alpha \, \mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \lesssim |t|^{-m\mu+\frac{(1-\delta)(1+\varepsilon)}{\delta}} + |t|^{\mu+\frac{(1-\delta)(2+\varepsilon)}{\delta}-1} + |t|^{m\mu-1+\frac{(1-\delta)(1+\varepsilon)}{\delta}}$$
$$+ |t|^{\upsilon_0+\mu+\varepsilon+\frac{(1-\delta)(1+\varepsilon)}{\delta}-1} + |t|^{\rho_0+\mu+\frac{(1-\delta)(1+\varepsilon)}{\delta}-1}.$$

Then the proposition follows with $\varepsilon < \frac{\delta\sigma}{2}$ and $\mu = \frac{1}{2m}$ with m sufficiently large, so that holds $\mu < \frac{\sigma}{2}$. □

Now consider the case $\frac{\pi}{2} \in [a, b]$. We have to exclude this critical point $\frac{\pi}{2}$. For this we denote the interval $\left(\frac{\pi}{2} - \gamma, \frac{\pi}{2} + \gamma\right)$ with J_γ (γ from $\left(0, \frac{\pi}{2}\right)$ is still to be chosen) and use the easy estimate

$$\left|\int_a^b hC_j^\alpha \, \mathrm{e}^{\mathrm{i}K_j^\alpha}\right| \leqslant C\gamma \sup_{x\in(a,b)} \left|C_j^\alpha\right| + \left|\int_{[a,b]\setminus J_\gamma} hC_j^\alpha \, \mathrm{e}^{\mathrm{i}K_j^\alpha}\right|.$$

The set $[a, b]\setminus J_\gamma$ is the union of at most two closed intervals, which do not contain the point $\frac{\pi}{2}$. We can therefore use for these intervals the previous results. We have only to note that the following inequality holds:

$$\inf_{x\in[a,b]\setminus J_\gamma} |\cos x| \geqslant \cos\left(\frac{\pi}{2} - \gamma\right) = \sin\gamma.$$

So we obtain instead of (50) the estimate

$$\left|\int_a^b hC_j^\alpha \, \mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \leqslant C(\gamma + \varepsilon_m) \sup_{x\in(a,b)} \left|C_j^\alpha\right| + C\left(\sum_{k=1}^{m-1} \frac{\varepsilon_{k-1}}{|t|\varepsilon_k \sin\gamma} \sup_{\substack{x\in(a,b)\\ \delta=0,1}} \left|\left(C_j^\alpha\right)^{(\delta)}\right|\right)$$
$$+ \sum_{k=1}^{m-1} \frac{C}{\varepsilon_k \sin\gamma}\left(\frac{2\varepsilon_k \sin\gamma + \varepsilon_{k-1}}{\varepsilon_k |t| \sin\gamma} + \frac{\varepsilon_{k-1} x_{j-1}^{\frac{2-\delta}{\delta}}}{|t|^{1-\upsilon_0}} + \frac{\varepsilon_{k-1}}{|t|^{1-\rho_0}}\right) \sup_{x\in(a,b)} \left|C_j^\alpha\right|. \qquad (51)$$

Proposition 6.2. *Suppose $j = m(t) + 1$. Let $[a, b]$ be an arbitrary closed subinterval of $(0, \pi)$. Then for each $\varepsilon > 0$, $\varpi > 0$ there exist $t_0 > 0$ and a constant C, so that for $|t| > t_0$ holds*

$$\sum_{\alpha\in A_{2,j}} \left|\int_a^b hC_j^\alpha \, \mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \leqslant C|t|^{-\min\{\varpi, \frac{1-2\varpi}{2}, \frac{2\delta-1}{\delta}-\varpi, 1-\upsilon_0-\varpi, 1-\rho_0-\varpi\}+\frac{(1-\delta)}{\delta}+\varepsilon}.$$

Proof. For small $\gamma > 0$ holds the inequality $\sin\gamma \geqslant \frac{\gamma}{2}$. With $\gamma = |t|^{-\varpi}$ we have then from (51) the estimate

$$\sum_{\alpha\in A_{2,j}} \left|\int_a^b hC_j^\alpha \,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \lesssim \left((|t|^{-\varpi} + |t|^{-m\mu})x_j^{\frac{(1-\delta)(1+\varepsilon)}{\delta}} + \frac{m}{|t|^{1-\mu-\varpi}} x_j^{\frac{(1-\delta)(2+\varepsilon)}{\delta}}\right)$$
$$+ \left(\sum_{k=1}^{m-1}\left(\frac{3}{|t|^{1-(k+1)\mu-2\varpi}}\right) + \frac{m x_{j-1}^{\frac{2-\delta}{\delta}}}{|t|^{1-\upsilon_0-\mu-\varpi}} + \frac{m|t|^{\rho_0}}{|t|^{1-\mu-\varpi}}\right) x_j^{\frac{(1-\delta)(1+\varepsilon)}{\delta}}.$$

Using $j = m(t) + 1$ and (12), we continue this as follows:

$$\sum_{\alpha\in A_{2,j}} \left|\int_a^b hC_j^\alpha \,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \lesssim \left(|t|^{-\varpi} + |t|^{-m\mu} + |t|^{\mu+\frac{1-\delta}{\delta}+\varpi-1}\right)|t|^{\frac{(1-\delta)(1+\varepsilon)}{\delta}}$$
$$+ (|t|^{m\mu-1+2\varpi} + |t|^{\upsilon_0+\mu+\varpi+\varepsilon-1} + |t|^{\rho_0+\mu+\varpi-1})|t|^{\frac{(1-\delta)(1+\varepsilon)}{\delta}}.$$

Then the proposition follows with $\mu = \frac{1-2\varpi}{2m}$ and m sufficiently large. □

7. The terms with large $\max|\alpha_r|$

The heading of this section refers to those terms from (28), which are not considered in the previous two sections, that is to the terms, which correspond to such α from $\mathbb{Z}^j$, for which there exists r_0, such that $|\alpha_{r_0}| > \gamma_{r_0}^j$.

We use in this section an easy estimate

$$\left|\int_a^b h(x)C_j^\alpha(x)\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \leqslant C\prod_{r=1}^{j} p_{0,r}^{-|\alpha_r|}. \tag{52}$$

($p_{0,r}$ is defined as in section 3, that is by $\frac{\sqrt{v_r^2+4a_0^2}}{v_r}$.) In the case $j > m(t)+1$ we have then the estimates

$$\left|\sum_{\alpha\in A_{3,j}}\int_a^b h(x)C_j^\alpha(x)\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \leqslant \left(\sum_{r_0=1}^{j-1}\sum_{\{\alpha\in\mathbb{Z}^j \| \alpha_{r_0}|\geqslant\gamma_{r_0}^j\}} C\prod_{r=1}^{j} p_{0,r}^{-|\alpha_r|}\right)$$
$$\leqslant C(j-1)\left(p_{0,j}^{-\min_{r=1,\ldots,j-1}(\gamma_r^j)}\prod_{r=1}^{j}\left(\sum_{k=-\infty}^{+\infty} p_{0,r}^{-k}\right)\right)$$
$$\leqslant jC\left(p_{0,j}^{-\min_{r=1,\ldots,j-1}(\gamma_r^j)}\prod_{r=1}^{j}\left(\frac{p_{0,r}+1}{p_{0,r}-1}\right)\right). \tag{53}$$

In cases $j = m(t)$ and $j = m(t)+1$ the following estimate is similarly obtained:

$$\left|\sum_{\alpha\in A_{3,j}}\int_a^b h(x)C_j^\alpha(x)\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \leqslant jC\left(p_{0,j}^{-\min_{r=1,\ldots,j}(\gamma_r^j)}\prod_{r=1}^{j}\left(\frac{p_{0,r}+1}{p_{0,r}-1}\right)\right). \tag{54}$$

We consider for the moment a little more general situation.

Lemma 7.1. *Let (μ_j) be an arbitrary sequence of positive real numbers. Then there exists a constant C, such that the following estimate holds:*

$$j\left(p_{0,j}^{-\mu_j}\prod_{r=1}^{j}\left(\frac{p_{0,r}+1}{p_{0,r}-1}\right)\right) \leqslant C\,\mathrm{e}^{-\mu_j\frac{2a_0^2}{v_j^2}}v_j^3.$$

Proof. We have from $\lim_{j\to+\infty} v_j = +\infty$ (which is the consequence of (3)) the relation $\lim_{j\to\infty}\left(1+\frac{4a_0^2}{v_j^2}\right)^{\frac{v_j^2}{4a_0^2}} = e$. Therefore, there exists a constant $\tilde{C}_1$, such that holds

$$p_{0,j}^{-\mu_j} = \left(\frac{v_r}{\sqrt{v_r^2+4a_0^2}}\right)^{\mu_j} = \left(\left(\frac{1}{1+\frac{4a_0^2}{v_j^2}}\right)^{\frac{v_j^2}{4a_0^2}}\right)^{\frac{1}{2}\mu_j\frac{4a_0^2}{v_j^2}} \leqslant \tilde{C}_1\,\mathrm{e}^{-\frac{2\mu_j a_0^2}{v_j^2}}.$$

From (3) follows $\lim\limits_{j\to\infty} j a_0^{-2j} v_j^{-1}\prod_{r=1}^{j-1}\left(v_r^2+a_0^2\right) = 0$. Particularly, there exists a constant $\tilde{C}_2$, such that

$$j\left(\prod_{r=1}^{j}\left(\frac{p_{0,r}+1}{p_{0,r}-1}\right)\right) = j\left(\prod_{r=1}^{j}\frac{(p_{0,r}+1)^2}{p_{0,r}^2-1}\right) = j\left(\prod_{r=1}^{j}\frac{\left(\sqrt{v_r^2+4a_0^2}+v_r\right)^2}{4a_0^2}\right)$$

$$\leqslant j\left(\prod_{r=1}^{j}\frac{v_r^2+4a_0^2}{a_0^2}\right) \leqslant \tilde{C}_2 v_j\left(v_j^2+4a_0^2\right) \leqslant 2\tilde{C}_2 v_j^3.$$

With $C = 2\tilde{C}_1\tilde{C}_2$we obtain the desired inequality. □

We are now able to prove the main result of the present section.

Proposition 7.2. *Let ε be any positive number and $\delta \in \left(\frac{1}{2},1\right)$. Denote with $G(t,\delta,\rho_0,\upsilon_0,\varepsilon)$ the value $|t|^{\min\{1-3\varepsilon,1+\upsilon_0-\varepsilon-\frac{1}{\delta},1+\rho_0-2\varepsilon-\frac{1}{\delta}\}}$. Then there exists a constant C, such that holds*

$$\sum_{j=m(t)}^{\infty}\sum_{\alpha\in A_{3,j}}\left|\int_a^b h(x)C_j^\alpha(x)\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \leqslant C\mathrm{e}^{-2a_0^2 G(t,\delta,\rho_0,\upsilon_0,\varepsilon)}.$$

Proof. For the value $\min\left(\gamma_r^j\right)$ we have for large j,t with (3) and (12) in the case $j > m(t)+1$:

$$\min_{r=1,\ldots,j-1}\left(\gamma_r^j\right) = x_j\left(4x_{j-1}^{\frac{2-\delta}{\delta}}\right)^{-1} \geqslant x_j^{1-\varepsilon}$$

in the case $j = m(t)$:

$$\min_{r=1,\ldots,j}\left(\gamma_r^j\right) = |t|\left(2x_j^{\frac{2-\delta}{\delta}}\right)^{-1} \geqslant |t|^{1-\varepsilon}$$

and in the case $j = m(t)+1$:

$$\min_{r=1,\ldots,j}(\gamma_r^j) = \min\left\{|t|^{\upsilon_0}, \left(2^{-1}|t|x_{j-1}^{\frac{\delta-2}{\delta}}\right)^{\rho_0}\right\} \geqslant |t|^{\min\{\upsilon_0,\rho_0-\varepsilon\}}.$$

Then we have from (53) and (54), using lemma 7.1 (with $\mu_j = \min\gamma_r^j$) the estimates

$$\sum_{\alpha\in A_{3,j}}\left|\int_a^b hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \lesssim x_j^{\frac{3(1-\delta)}{2\delta}}\,\mathrm{e}^{-2a_0^2 x_j^{2-\varepsilon-1/\delta}} \qquad j > m(t)+1$$

$$\sum_{\alpha\in A_{3,j}}\left|\int_a^b hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \lesssim x_j^{\frac{3(1-\delta)}{2\delta}}\,\mathrm{e}^{-2a_0^2|t|^{1-\varepsilon}x_j^{1-1/\delta}} \qquad j = m(t)$$

$$\sum_{\alpha\in A_{3,j}}\left|\int_a^b hC_j^\alpha\,\mathrm{e}^{\mathrm{i}K_j^\alpha}\,\mathrm{d}x\right| \lesssim x_j^{\frac{3(1-\delta)}{2\delta}}\,\mathrm{e}^{-2a_0^2|t|^{\min\{\upsilon_0,\rho_0-\varepsilon\}}x_j^{1-1/\delta}} \qquad j = m(t)+1.$$

From the condition $\delta \in \left(\frac{1}{2}, 1\right)$ follows $2 - 1/\delta > 0$. We can conclude then, because of rapid growth of x_j (condition (3)), that the series

$$\sum_{j=m(t)+2}^{\infty} \sum_{\alpha \in A_{3,j}} \left| \int_a^b hC_j^\alpha \, \mathrm{e}^{\mathrm{i}K_j^\alpha} \, \mathrm{d}x \right|$$

is negligently small in comparison with

$$\sum_{\alpha \in A_{3,m(t)}} \left| \int_a^b hC_{m(t)}^\alpha \, \mathrm{e}^{\mathrm{i}K_{m(t)}^\alpha} \mathrm{d}x \right| + \sum_{\alpha \in A_{3,m(t)+1}} \left| \int_a^b hC_{m(t)+1}^\alpha \, \mathrm{e}^{\mathrm{i}K_{m(t)+1}^\alpha} \, \mathrm{d}x \right|.$$

We estimate the last expression using the estimates from above and (12) as follows:

$$\sum_{j=m(t)}^{m(t)+1} \sum_{\alpha \in A_{3,j}} \left| \int_a^b hC_j^\alpha \, \mathrm{e}^{\mathrm{i}K_j^\alpha} \, \mathrm{d}x \right| \lesssim \frac{x_m(t)^{\frac{3(1-\delta)}{2\delta}}}{\mathrm{e}^{2a_0^2|t|^{1-2\varepsilon}}} + \frac{|t|^{\frac{3(1-\delta)}{2\delta}}}{\mathrm{e}^{2a_0^2|t|^{\min\{\upsilon_0, \rho_0 - \varepsilon\}+1-1/\delta}}}.$$

We have for $|t|$ sufficiently large the inequality $|t|^{\frac{3(1-\delta)}{2\delta}} \leqslant \mathrm{e}^{2a_0^2|t|^\varepsilon}$, from which then the proposition follows with (12). □

8. Proof of theorem 1.2

(i) We have now to specify the values ρ_0 and υ_0. The smallest values, for which proposition 7.2 gives us the desired estimate, are $\rho_0 = \upsilon_0 = \frac{1}{\delta} - 1 + \sigma$ with $\sigma > 0$ (for all smaller values holds $G(t, \delta, \rho_0, \upsilon_0, \varepsilon) \leqslant 0$, which stays in contradiction with the relation (1)). So we set $\rho_0 = \upsilon_0 = \frac{1}{\delta} - 1 + \sigma$ with $\sigma > 0$ still to be specified. Then we can conclude from proposition 7.2 that the contribution of the 'terms with large $\max |\alpha_r|$' in (9) can be estimated by $\mathrm{e}^{-t^{\sigma/2}}$ (if $\varepsilon > 0$ is chosen sufficiently small).

As for non-resonant terms, we see then from propositions 5.2, 5.3 and 5.4 that we can estimate the sum of these terms by $C|t|^{-m}$ with arbitrary m (we use (3) to conclude that the contribution of integrals $\int_a^b hC_j^\alpha \, \mathrm{e}^{\mathrm{i}K_j^\alpha} \, \mathrm{d}x$ for $j > m(t) + 1$ is smaller than the contribution of these integrals with $j = m(t)$ and $j = m(t) + 1$).

So the crucial contribution comes from the resonant terms. Proposition 6.1 implies that the sum of these terms can be estimated by $C|t|^{-\min\{\frac{3}{2} - \frac{1}{\delta} - \sigma, 3 - \frac{2}{\delta} - 2\sigma\}}$. In the case $\delta > \frac{2}{3}$ the last expression takes the form $C|t|^{\frac{1}{\delta} - \frac{3}{2} + \sigma}$. Thus (i) is proved.

(ii) We set again $\rho_0 = \upsilon_0 = \frac{1}{\delta} - 1 + \sigma$. Everything said about the non-resonant terms and the 'terms with large $\max |\alpha_r|$' remains valid in this case without a change. The crucial contribution comes again from the resonant terms. We have now only to specify the value ϖ. From proposition 6.2 we have now the estimate

$$\sum_{\alpha \in A_{2,m(t)+1}} \left| \int_a^b hC_{m(t)+1}^\alpha \, \mathrm{e}^{\mathrm{i}K_{m(t)+1}^\alpha} \, \mathrm{d}x \right| \leqslant C|t|^{-\min\{\varpi, \frac{1-2\varpi}{2}, 2 - \frac{1}{\delta} - \varpi - \sigma\} + \frac{(1-\delta)}{\delta} + \sigma}.$$

So we have to choose ϖ so that $\min\left\{\varpi, \frac{1-2\varpi}{2}, 2 - \frac{1}{\delta} - \varpi - \sigma\right\}$ takes the largest value that is obtained (if $\delta > 2/3$) by $\varpi = 1/4$. We have then the estimate

$$\sum_{\alpha \in A_{2,m(t)+1}} \left| \int_a^b hC_{m(t)+1}^\alpha \, \mathrm{e}^{\mathrm{i}K_{m(t)+1}^\alpha} \, \mathrm{d}x \right| \leqslant C|t|^{-\frac{5}{4} + \frac{1}{\delta} + \sigma}$$

from which statement (ii) follows.

(iii) We consider the case $|t| \notin \mathcal{R}$ and prove that in this case (for sufficiently large value of $|t|$) the set $A_{2,m(t)+1}$ of resonant terms is empty, so the better estimate (of the order of $C|t|^{-m}$

with arbitrary m) is possible. $|t| \notin \mathcal{R}$ implies $x_j^{\frac{\delta}{2\delta-1}+\varepsilon} < |t| < x_{j+1}/2$ with some j. This j must by the definition of $m(t)$ be equal to $m(t)+1$. We have now to show that for all α with $|\alpha_r| \leqslant \gamma_r^{m(t)+1}$ holds (29). Similarly to (39) we obtain

$$\min_{x\in[a,b]}\left|t\sin(x) + \sum_{r=1}^{j}\alpha_r x_r\right| \geqslant |t|a_0 - |\alpha_{m(t)+1}|x_{m(t)+1} - \left(|t|x_{j-1}^{\frac{\delta-2}{\delta}}\right)^{\rho_0}\sum_{r=1}^{j-1}x_r$$
$$\geqslant |t|a_0 - |t|^{\upsilon_0}x_{m(t)+1} - |t|^{\rho_0}x_{m(t)+1}^{\sigma}$$

for large $|t|$ and small $\sigma > 0$. We set as in (i) and (ii) $\rho_0 = \upsilon_0 = \frac{1}{\delta} - 1 + \sigma$. So we can continue the previous inequality as follows:

$$\min_{x\in[a,b]}\left|t\sin(x) + \sum_{r=1}^{j}\alpha_r x_r\right| \geqslant |t|a_0 - 2|t|^{\frac{1}{\delta}-1+\sigma}x_{m(t)+1}$$
$$= |t|^{\frac{1}{\delta}-1+\sigma}\left(|t|^{\frac{2\delta-1}{\delta}-\sigma}a_0 - 2x_{m(t)+1}\right). \tag{55}$$

From $|t| > x_{m(t)+1}^{\frac{\delta}{2\delta-1}+\varepsilon}$ follows for sufficiently small σ the inequality $|t|^{\frac{2\delta-1}{\delta}-\sigma} > x_{m(t)+1}^{1+\frac{\varepsilon}{2}}$. Relation (29) follows now from (55), using (3). Thus $A_{2,m(t)+1}$ is empty and the estimate

$$|(f\,\mathrm{d}\rho)\widehat{\ }(t)| \leqslant C|t|^{-m}$$

holds with some C for arbitrary m. □

9. Proof of theorem 1.3

If the sequence (v_n) from (2) is bounded, there exists a constant $A > 1$, such that for all r holds the inequality $p_{0,r} \geqslant A$, from which follows with $B = 1 + \frac{2}{\sqrt{A}-1}$ the inequality

$$\frac{p_{0,r}+1}{p_{0,r}-1} < \frac{\sqrt{p}_{0,r}+1}{\sqrt{p}_{0,r}-1} \leqslant B.$$

We have then for the derivatives of coefficients from (13) the estimate

$$\sum_{\alpha\in\mathbb{Z}^{m(t)+1}}\ \sup_{\substack{\kappa=0,\ldots,k\\ x\in(a,b)}}\left|\left(C_{m(t)+1}^{\alpha}(x)\right)^{(\kappa)}\right| \leqslant \tilde{P}_k B^{m(t)+1} \tag{56}$$

where $\tilde{P}_k$ are possibly different from $\tilde{P}_k$ from (13).

We can use in the case of a bounded potential the inequalities from corollary 4.3 with $\delta = 1$ (for the proof see [4]). We set correspondingly $\delta = 1$ in the definition of the values γ_r^j.

It is easy to see that all results from section 5 remain valid for bounded potentials also, because estimate (56) is even better than estimate (37). Therefore, the contribution of non-resonant terms is in the case of a bounded potential also of the order of $C|t|^{-m}$. So the only changes we have to make concern estimates from sections 6 and 7.

We start with the modifications, which we have to carry out in section 7. We replace estimates (53) and (54) by the estimate

$$\left|\sum_{\alpha\in A_{3,j}}\int_a^b h(x)C_j^{\alpha}(x)\,\mathrm{e}^{\mathrm{i}K_j^{\alpha}}\,\mathrm{d}x\right| \leqslant jCA^{-\min(\gamma_r^j)}B^j.$$

Then the final estimate for the terms with large $\max|\alpha_r|$ reads

$$\sum_{j=m(t)}^{\infty}\sum_{\alpha\in A_{3,j}}\left|\int_a^b hC_j^{\alpha}\,\mathrm{e}^{\mathrm{i}K_j^{\alpha}}\,\mathrm{d}x\right| \leqslant CA^{-|t|^{\min\{\upsilon_0,\rho_0\}-2\varepsilon}}$$

where we have used condition (4) to obtain

$$\lim_{t\to\pm\infty}(m(t)+1)B^{m(t)+1}A^{-|t|^{\varepsilon}}=0 \qquad \text{for all} \quad \varepsilon>0. \tag{57}$$

So we can conclude that the contribution of these terms to (9) is for any choice of $\upsilon_0 \in (0,1)$ and $\rho_0 \in (0,1)$ smaller than the contribution of non-resonant terms (we have only to set in the last inequality $\varepsilon = \min\{\upsilon_0, \rho_0\}/3$).

The crucial contribution comes (as in the case of an unbounded potential) again from the resonant terms. We start our consideration here with inequality (51) and use inequality (56) to obtain

$$\sum_{\alpha\in A_{2,j}}\left|\int_a^b hC_j^{\alpha}\,\mathrm{e}^{\mathrm{i}K_j^{\alpha}}\,\mathrm{d}x\right| \lesssim B^j\left(\gamma+\varepsilon_m+\sum_{k=1}^{m-1}\frac{\varepsilon_{k-1}}{|t|\varepsilon_k\sin\gamma}\right)$$
$$+\sum_{k=1}^{m-1}\frac{1}{\varepsilon_k\sin\gamma}\left(\frac{2\varepsilon_k\sin\gamma+\varepsilon_{k-1}}{\varepsilon_k|t|\sin\gamma}+\frac{\varepsilon_{k-1}x_{j-1}^{\frac{2-\delta}{\delta}}}{|t|^{1-\upsilon_0}}+\frac{\varepsilon_{k-1}}{|t|^{1-\rho_0}}\right)B^j$$

where j stays for $m(t)+1$. We set $\varepsilon_k = |t|^{-k\mu}$ with $\mu = \frac{1-2\varpi}{2m}$, $\gamma = |t|^{-\varpi}$ and $\upsilon_0 = \rho_0 = \tau > 0$ to obtain from the last inequality the following estimate:

$$\sum_{\alpha\in A_{2,m(t)+1}}\left|\int_a^b hC_{m(t)+1}^{\alpha}\,\mathrm{e}^{\mathrm{i}K_{m(t)+1}^{\alpha}}\,\mathrm{d}x\right| \lesssim \left(|t|^{-\varpi}+|t|^{\frac{2\varpi-1}{2}}+|t|^{\tau+\varpi-1}\right)|t|^{\sigma}B^{m(t)+1}.$$

We have now only to set $\varpi = 1/4$ and then to choose any τ from $(0, 1/4)$, because we have for all $\varepsilon > 0$ from condition (4) the relation

$$\sup_{t\in\mathbb{R}\setminus[-1,1]} B^{m(t)+1}|t|^{-\varepsilon} < \infty.$$

□

Acknowledgments

I thank H Behncke and C Remling for constant encouragement and stimulating discussions.

References

[1] Jitomirskaya S and Last Y 1999 Power-law subordinacy and singular spectra: I. Half-line operators *Acta Math.* **183** 171–89
[2] Jitomirskaya S and Last Y 1996 Dimensional Hausdorff properties of singular continuous spectra *Phys. Rev. Lett.* **76** 1765–9
[3] Kiselev A, Last Y and Simon B 1998 Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators *Commun. Math. Phys.* **194** 1–45
[4] Krutikov D and Remling C 2001 Schrödinger operators with sparse potentials: asymptotics of the Fourier transform of the spectral measure *Commun. Math. Phys.* **223** 509–32
[5] Last Y 1996 Quantum dynamics and decompositions of singular continuous spectra *J. Funct. Anal.* **142** 406–45
[6] Lyons R 1985 Fourier–Stieltjes coefficients and asymptotic distribution modulo 1 *Ann. Math.* **122** 155–70
[7] Pearson D B 1978 Singular continuous measures in scattering theory *Commun. Math. Phys.* **60** 13–36
[8] Pearson D B 1993 Value distribution and spectral analysis of differential operators *J. Phys. A: Math. Gen.* **26** 4067–80
[9] Simon B and Stolz G 1996 Operators with singular continuous spectrum: V. Sparse potentials *Proc. Am. Math. Soc.* **124** 2073–80